

# Solutions of the Fully Compressible Semi-Geostrophic System

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## Abstract

The fully compressible semi-geostrophic system is widely used in the modelling of large-scale atmospheric flows. In this paper, we prove rigorously the existence of weak Lagrangian solutions of this system, formulated in the original physical coordinates. In addition, we provide an alternative proof of the earlier result on the existence of weak solutions of this system expressed in the so-called geostrophic, or dual, coordinates. The proofs are based on the optimal transport formulation of the problem and on recent general results concerning transport problems posed in the Wasserstein space of probability measures.

## 1 Introduction

The behaviour of the atmosphere is governed by the compressible Navier-Stokes equations, together with the laws of thermodynamics, equations describing phase changes, source terms and boundary fluxes. These equations are too complex to be solved accurately in a large-scale atmospheric context and therefore reductions and approximations of the Navier-Stokes equations are often used to validate and understand the solutions that have been computed.

One such approximation, valid on scales where the effects of rotation dominate the flow, is the system of semi-geostrophic equations, see equations (3.1)-(3.5) below. First introduced by Eliassen [9] and then rediscovered by Hoskins [11], the semi-geostrophic equations are an extremely useful model, particularly in describing the formation of fronts, and are both studied theoretically and used widely for numerical modelling and simulations.

In this paper, we prove rigorously the existence of weak Lagrangian solutions of the fully compressible semi-geostrophic system. The proof combines ideas of several previous papers, starting with the pioneering work of Benamou and Brenier [4] on the formulation of the semi-geostrophic equations as an optimal transport problem, and modern methods in the analysis of Hamiltonian ODEs in probability spaces, in particular the work of Ambrosio [2], [3].

For an accurate representation of the behaviour of large-scale atmospheric flow, one should consider the fully compressible semi-geostrophic equations with variable Coriolis parameter and a free surface condition. The complexity of this problem means that so far results are obtained after relaxing one or more of these conditions.

In [4], Benamou and Brenier assumed the fluid to be incompressible, the Coriolis parameter  $f$  constant and the boundaries rigid. They then used a change of variables, introduced

by Hoskins in [11], to derive the so-called dual formulation and to prove that stable weak solutions of the semi-geostrophic equations in these dual variables exist, by interpreting the equations as a coupled Monge-Ampère transport problem.

In [7], Cullen and Gangbo relaxed the assumption of rigid boundaries with a more physically appropriate free boundary condition. However, they additionally assumed constant potential temperature to obtain the 2-D system known as the semi-geostrophic shallow water system. After passing to dual variables, they showed existence of stable weak solutions for this system of equations.

In [8], Cullen and Maroofi proved that stable weak solutions of the 3-D compressible semi-geostrophic system in its dual formulation exist, returning to the assumption of rigid boundaries.

The main problem posed by the existence results in [4], [7] and [8] is that they are all proved in dual space. It is difficult to relate these results directly to the Navier-Stokes equations, or indeed other reductions of them. For this reason, and in order to give the dual space results physical meaning, Cullen and Feldman [6] mapped the solutions back to the original, physical coordinates and extended the results of [4] and [7], proving existence of weak Lagrangian solutions in physical space, in the incompressible case.

In this paper, we utilise recent results in the analysis of ODEs in spaces of measures, in particular those of [2], [3], in order to provide an alternative proof of the dual space result in [8]. We also extend considerably the results of [8] to prove the existence of weak Lagrangian solutions of the fully compressible semi-geostrophic system in the original physical coordinates. This is based on the existence of an appropriate flow map with rather low regularity; however we also show that, if we could assume additional regularity, then the solutions derived would determine classical solutions.

The paper is organised as follows: in Section 2 we introduce various definitions, and the notation to be used throughout; in Section 3 we summarise the existing results regarding existence of solutions in dual space that are necessary for the theory presented in the following sections; in Section 4 we use the results of [2] to provide an alternative proof of the existence of solutions in dual space; in Section 5 we define the concept of a weak Lagrangian solution and formulate our main theorem; in Section 6 we prove the existence and important properties of a Lagrangian flow in dual space; finally, in Section 7 we map the dual space Lagrangian flow to physical space and use it to prove our main result, namely the existence of a weak Lagrangian solution.

## 2 Useful Conventions, Notation and Definitions

### Physical quantities and constants (2.1)

- (i)  $\Omega$  denotes an open bounded convex set in  $\mathbb{R}^3$ , representing the physical domain containing the fluid;  $\tau > 0$  is a fixed positive constant; all functions in physical coordinates are defined for  $(t, \mathbf{x}) \in [0, \tau) \times \Omega$ ;
- (ii)  $\mathbf{u}(t, \mathbf{x})$  represents the 3-D velocity of the fluid;
- (iii)  $\mathbf{u}^g(t, \mathbf{x}) = (u_1^g(t, \mathbf{x}), u_2^g(t, \mathbf{x}), 0)$  represents the geostrophic velocity;
- (iv)  $p(t, \mathbf{x})$  represents the pressure;
- (v)  $\rho(t, \mathbf{x})$  represents the density;

- (vi)  $\theta(t, \mathbf{x})$  represents the potential temperature. Given its physical meaning, we assume  $\theta(t, \mathbf{x})$  to be strictly positive and bounded;
- (vii)  $\phi(\mathbf{x})$  is the given geopotential representing gravitation and centrifugal forces. We assume that  $\phi \in C^2(\bar{\Omega})$  and that  $\frac{\partial}{\partial x_3}\phi(\mathbf{x}) \neq 0$  for all  $\mathbf{x} \in \bar{\Omega}$ ;
- (viii)  $f_{\text{cor}}$  denotes the Coriolis parameter, which we assume to be constant;
- (ix)  $p_{\text{ref}}$  is the reference value of the pressure;
- (x)  $c_v$  is a constant representing the specific heat at a constant pressure;
- (xi)  $c_p$  is a constant representing the specific heat at a constant volume;
- (xii)  $R$  represents the gas constant and satisfies  $R = c_p - c_v$ ;
- (xiii)  $\kappa = c_p/c_v$  denotes the ratio of specific heats (this is approximately 1.4 for air).

### Notations and other conventions (2.2)

- Throughout, we will only consider measures that are absolutely continuous with respect to Lebesgue measure. Given an open set  $A$  in  $\mathbb{R}^3$ , we will denote by
  - $P_{ac}(A)$  - the set of probability measures in  $\mathbb{R}^3$  with supports contained in  $A$ ;
  - $\chi_A$  - the characteristic function of  $A$ .
- Unless otherwise specified, measurable means Lebesgue measurable and *a.e.* means Lebesgue-*a.e.*
- $D_t$  denotes the Lagrangian derivative, defined as  $D_t = \partial_t + \mathbf{u} \cdot \nabla$ .
- The unit vector  $(0, 0, 1)$  in  $\mathbb{R}^3$  is denoted by  $\mathbf{e}_3$ .
- For convenience, we will sometimes use the notation  $F_{(t)}(\cdot) = F(t, \cdot)$  to denote the map  $F$  evaluated at fixed time  $t$ . In fact, we will often leave out time  $t$  as an argument.

### Important definitions

**Definition 2.1.** *We define*

$$P_{ac}^2(\mathbb{R}^3) := \{\mu \in P_{ac}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\mathbf{x}|^2 \mu(\mathbf{x}) d\mathbf{x} < +\infty\},$$

*with tangent space*

$$T_\mu P_{ac}^2(\mathbb{R}^3) = \overline{\{\nabla \varphi : \varphi \in C_c^\infty(\mathbb{R}^3)\}}^{L^2(\mu; \mathbb{R}^3)}. \quad (2.3)$$

**Definition 2.2.** *Let  $\gamma$  be a probability measure on the product space  $\mathbb{R}^3 \times \mathbb{R}^3$ . Then, Borel probability densities  $\mu_1$  and  $\mu_2$  in  $\mathbb{R}^3$  are called marginals of  $\gamma$  if*

$$\mu_1[A] = \gamma[A \times \mathbb{R}^3], \quad \mu_2[B] = \gamma[\mathbb{R}^3 \times B]$$

*for all measurable  $A, B \subset \mathbb{R}^3$ . We define*

$$\Gamma(\mu_1, \mu_2) = \{\gamma : \gamma \text{ probability measure on } \mathbb{R}^3 \times \mathbb{R}^3 \text{ with marginals } \mu_1 \text{ and } \mu_2\}.$$

**Definition 2.3.** Given two Borel probability densities  $\mu_1(\cdot)$  and  $\mu_2(\cdot)$  in  $\mathbb{R}^3$ , we define the Wasserstein-2 distance,  $W_2$ , between  $\mu_1$  and  $\mu_2$  as follows:

$$W_2^2(\mu_1, \mu_2) := \inf_{\gamma \in \Gamma(\mu_1, \mu_2)} \int_{\mathbb{R}^3 \times \mathbb{R}^3} |\mathbf{x} - \mathbf{y}|^2 d\gamma(\mathbf{x}, \mathbf{y}). \quad (2.4)$$

We denote by  $\Gamma_0(\mu_1, \mu_2)$  the set of minimisers of (2.4).

The Wasserstein distance indeed defines a distance in the space of probability measures on  $\mathbb{R}^3$  (or more generally, any complete separable metric space); it can be used as an optimal transport cost between these measures, see [14].

**Definition 2.4.** Let  $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be a measurable map. Let  $\mu_1$  be a measure in  $\mathbb{R}^3$ . Then we say that the measure  $\mu_2$  on  $\mathbb{R}^3$  is the push forward of  $\mu_1$  with respect to  $F$ , denoted  $F\#\mu_1 = \mu_2$ , if

$$\mu_2[A] = \mu_1[F^{-1}(A)]$$

for all measurable  $A \subset \mathbb{R}^3$ . If, in addition,  $\mu_1 \in L^1(\mathbb{R}^3)$ ,  $\mu_2 \in L^1(\mathbb{R}^3)$  and  $F\#\mu_1 = \mu_2$ , then (see [6, Corollary A.3])

$$\int_{\mathbb{R}^3} \varphi(F(\mathbf{x})) \mu_1(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^3} \varphi(\mathbf{X}) \mu_2(\mathbf{X}) d\mathbf{X}, \quad \forall \varphi \in L^\infty(\mathbb{R}^3). \quad (2.5)$$

**Remark 2.1.** Since we are only interested in measures that are absolutely continuous with respect to Lebesgue measure, we can guarantee existence of a unique minimiser  $\gamma_0$  in (2.4); see, for example, [3, Section 6.2.3]. We also have that  $\gamma_0 = (\mathbf{id}, \mathbf{R}_{\mu_1}^{\mu_2})\#\mu_1$  for some map  $\mathbf{R}_{\mu_1}^{\mu_2} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  which coincides  $\mu_1 - a.e.$  with the gradient of a convex function. Thus, the map  $\mathbf{R}_{\mu_1}^{\mu_2}$  is the unique minimiser of

$$\mathbf{R} \rightarrow \int_{\mathbb{R}^3} |\mathbf{x} - \mathbf{R}(\mathbf{x})|^2 \mu_1(\mathbf{x}) d\mathbf{x} \quad (2.6)$$

over all  $\mathbf{R}$  such that  $\mathbf{R}\#\mu_1 = \mu_2$ .

**Definition 2.5.** Given a set  $A \subset \mathbb{R}^3$  we say that a sequence  $(\mu_n) \subset P_{ac}(A)$  is narrowly convergent to  $\mu \in P_{ac}(A)$  as  $n \rightarrow \infty$  if

$$\lim_{n \rightarrow \infty} \int_A \varphi(x) \mu_n(x) dx = \int_A \varphi(x) \mu(x) dx$$

for every function  $\varphi \in C_b(A)$ , the space of continuous and bounded real functions defined on  $A$ .

**Definition 2.6.** Given a set  $A \subset \mathbb{R}^3$  and a function  $f : A \rightarrow \mathbb{R}$ , we say that  $f$  is semi-concave if there exists a constant  $\tilde{k} > 0$  such that the map  $x \rightarrow f(x) - \frac{1}{2}\tilde{k}^2|x|^2$  is concave in  $A$ . We call  $\tilde{k}$  a constant of semi-concavity. This definition can be generalised by replacing  $-|x|^2$  with any concave function.

Note that if  $g$  is semi-concave, then  $\nabla g$  has bounded variation.

### 3 The semi-geostrophic equations: formulation and existing results

The fully compressible semi-geostrophic equations posed in the domain  $[0, \tau) \times \Omega$ , with rigid boundaries  $\partial\Omega$ , are the following system of equations (see, for example, [8]):

$$D_t \mathbf{u}^g + f_{\text{cor}} \mathbf{e}_3 \times \mathbf{u} + \nabla \phi + \frac{1}{\rho} \nabla p = 0, \quad (3.1)$$

$$D_t \theta = 0, \quad (3.2)$$

$$D_t \frac{1}{\rho} = \frac{1}{\rho} \nabla \cdot \mathbf{u}, \quad (t, \mathbf{x}) \in [0, \tau) \times \Omega \quad (3.3)$$

$$f_{\text{cor}} \mathbf{e}_3 \times \mathbf{u}^g + \nabla \phi + \frac{1}{\rho} \nabla p = 0, \quad (3.4)$$

$$p = R\rho\theta \left( \frac{p}{p_{\text{ref}}} \right)^{\frac{\kappa-1}{\kappa}}; \quad (3.5)$$

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad (t, \mathbf{x}) \in [0, \tau) \times \partial\Omega. \quad (3.6)$$

The unknowns in the above equations are  $\mathbf{u}^g = (u_1^g, u_2^g, 0)$ ,  $\mathbf{u} = (u_1, u_2, u_3)$ ,  $p$ ,  $\rho$ ,  $\theta$ .

Equation (3.1) is the momentum equation; (3.2) represents the adiabatic assumption; (3.3) is the continuity equation and (3.4) represents hydrostatic and geostrophic balance. The equation (3.5) is the equation of state which relates the thermodynamic quantities to each other, and (3.6) is the rigid boundary condition, where  $\mathbf{n}$  is the outward normal to  $\partial\Omega$ .

The semi-geostrophic equations are a valid approximation when  $\frac{U}{f_{\text{cor}} L} \ll 1$ , and are accurate when  $\frac{H}{L} < \frac{f_{\text{cor}}}{N}$ , where  $U$  is a typical scale for horizontal speed;  $L$  is a typical horizontal scale;  $H$  is a typical vertical scale;  $N$  is the buoyancy frequency.

The energy associated with the flow, known as the *geostrophic energy*, is defined as

$$E(t) = \int_{\Omega} \left[ \frac{1}{2} |\mathbf{u}^g|^2(t, \mathbf{x}) + \phi(\mathbf{x}) + c_v \theta(t, \mathbf{x}) \left( \frac{p(t, \mathbf{x})}{p_{\text{ref}}} \right)^{\frac{\kappa-1}{\kappa}} \right] \rho(t, \mathbf{x}) d\mathbf{x}. \quad (3.7)$$

#### 3.1 Dual formulation

In [8], solutions were obtained using a transformation into the so-called geostrophic, or dual, coordinates  $\mathbf{y} = (y_1, y_2, y_3)$ . The coordinate transformation is given by:

$$\mathbf{T} : \Omega \rightarrow \Lambda \subset \mathbb{R}^3, \quad \mathbf{T}(t, \mathbf{x}) = (T_1(t, \mathbf{x}), T_2(t, \mathbf{x}), T_3(t, \mathbf{x})) = (y_1, y_2, y_3),$$

with

$$y_1 = x_1 + f_{\text{cor}}^{-1} u_2^g(t, \mathbf{x}), \quad y_2 = x_2 - f_{\text{cor}}^{-1} u_1^g(t, \mathbf{x}), \quad y_3 = \theta(t, \mathbf{x}). \quad (3.8)$$

Note that, by (2.1)(vi),  $\overline{\Lambda} \subset \mathbb{R}^2 \times [\delta, \frac{1}{\delta}]$  for some  $0 < \delta < 1$ .

Using this transformation, as well as (3.5), it was shown in [8] that we can write the energy in (3.7) as

$$\begin{aligned} E(t) &= \int_{\Omega} \frac{\frac{1}{2} f_{\text{cor}}^2 \{|x_1 - y_1|^2 + |x_2 - y_2|^2\} + \phi(\mathbf{x})}{y_3} \sigma(t, \mathbf{x}) d\mathbf{x} + K_1 \int_{\Omega} (\sigma(t, \mathbf{x}))^{\kappa} d\mathbf{x} \\ &= \int_{\Omega} \frac{\frac{1}{2} f_{\text{cor}}^2 \{|x_1 - T_1(t, \mathbf{x})|^2 + |x_2 - T_2(t, \mathbf{x})|^2\} + \phi(\mathbf{x})}{T_3(t, \mathbf{x})} \sigma(t, \mathbf{x}) d\mathbf{x} + K_1 \int_{\Omega} (\sigma(t, \mathbf{x}))^{\kappa} d\mathbf{x} \end{aligned} \quad (3.9)$$

where  $K_1 := c_v \left( \frac{R}{p_{\text{ref}}} \right)^{\kappa-1}$  is constant and

$$\sigma(t, \mathbf{x}) := \theta(t, \mathbf{x}) \rho(t, \mathbf{x}). \quad (3.10)$$

Using the results of [12], it was shown that stable solutions of (3.1)-(3.6) correspond to solutions that, at each fixed time  $t$ , minimise the energy  $E$  given by (3.9). We refer to this requirement as *Cullen's stability condition*.

This stability condition can be formulated in terms of optimal transport concepts. Indeed, define the potential density  $\nu := \mathbf{T} \# \sigma$  as the push forward of the measure  $\sigma$  under the map  $\mathbf{T}$  (see Definition 2.4). Then Cullen's stability condition is equivalent to the condition that, at each fixed time, the pair  $(\sigma, \mathbf{T})$  minimises the energy (3.9) amongst all pairs  $(\mu, \overline{\mathbf{T}})$  with  $\mu \in P_{ac}(\Omega)$  and  $\overline{\mathbf{T}} \# \mu = \nu$ . It was shown that, for a stable solution, the energy in (3.9) can be written, dropping the dependence on time, in the following form

$$E_\nu(\sigma) = \mathcal{E}(\sigma, \nu) + K_1 \int_{\Omega} (\sigma(t, \mathbf{x}))^\kappa d\mathbf{x}, \quad (3.11)$$

where

$$\mathcal{E}(\sigma, \nu) = \inf_{\overline{\mathbf{T}} \# \sigma = \nu} \int_{\Omega} c(\mathbf{x}, \overline{\mathbf{T}}(t, \mathbf{x})) \sigma(t, \mathbf{x}) d\mathbf{x} \quad (3.12)$$

with  $c$  a cost function, given by

$$c(\mathbf{x}, \mathbf{y}) = \frac{\frac{1}{2} f_{\text{cor}}^2 \{|x_1 - y_1|^2 + |x_2 - y_2|^2\} + \phi(\mathbf{x})}{y_3}. \quad (3.13)$$

The energy minimisation required by Cullen's stability condition has now been reformulated as an optimal transport problem. Cullen's stability condition amounts to the requirement that the change of variables  $\mathbf{T}(t, \cdot)$  from physical coordinates to geostrophic coordinates given by (3.8) is the optimal transport map between  $\sigma$  and  $\nu$ .

In [8], it is shown that, given a potential density  $\nu$ , there exists a unique minimiser  $\sigma$  of (3.11). Given this pair  $\nu$ , and  $\sigma$ , there always exists a unique minimiser  $\mathbf{T}$  in (3.12), given by the optimal map in the transport of  $\sigma$  to  $\nu$ ; the map  $\mathbf{T}$  admits a unique inverse  $\mathbf{T}^{-1}$ , which is the optimal map in the transport of  $\nu$  to  $\sigma$  with cost  $\tilde{c}(\mathbf{y}, \mathbf{x}) = c(\mathbf{x}, \mathbf{y})$ .

Using the stability condition and the transformation given by (3.8), the compressible semi-geostrophic equations (3.1)-(3.4) can be written in dual variables as follows [8]:

$$\partial_t \nu(t, \mathbf{y}) + \nabla \cdot (\nu(t, \mathbf{y}) \mathbf{w}(t, \mathbf{y})) = 0 \quad \text{in } [0, \tau) \times \Lambda, \quad (3.14)$$

$$\mathbf{w}(t, \mathbf{y}) = \mathbf{u}^g(t, \mathbf{S}(t, \mathbf{y})) = f_{\text{cor}} \mathbf{e}_3 \times [\mathbf{y} - \mathbf{S}(t, \mathbf{y})], \quad (3.15)$$

$$\mathbf{S}(t, \cdot) = \mathbf{T}^{-1}(t, \cdot), \quad (3.16)$$

$$\mathbf{T}(t, \cdot) \text{ is the unique optimal transport map in (3.12),} \quad (3.17)$$

$$\sigma(t, \cdot) \text{ minimises } E_{\nu(t, \cdot)}(\cdot) \text{ over } P_{ac}(\Omega), \quad (3.18)$$

$$\nu(0, \cdot) = \nu_0(\cdot) \in L^r(\Lambda_0), \quad r \in (1, \infty), \quad \Lambda_0 \subset \mathbb{R}^3 \text{ compact.} \quad (3.19)$$

Equation (3.14) is the continuity equation satisfied by the potential density; (3.15) defines the geostrophic velocity in dual variables; (3.16), (3.17) and (3.18) are required for a stable solution; and (3.19) is the prescribed initial condition.

The main result of [8] is the existence of a stable weak solution of the semi-geostrophic system in dual variables (3.14)-(3.19), proved by an approximation procedure. In the same paper, several important results concerning optimal transport are proved. In Section 3.2, we summarise these results.

**Remark 3.1.** Since  $\nu_0$  has compact support, for all  $t \in (0, \tau)$  the solution  $\nu(t, \mathbf{y})$  of the evolution (3.14) starting at time  $t = 0$  from  $\nu_0$  has compact support in  $\mathbb{R}^3$ . By (2.1)(vi),  $\text{supp}(\nu)$  is contained in a bounded open set  $\Lambda$ , which is dependent on  $\tau$ , such that  $\overline{\Lambda} \subset \mathbb{R}^2 \times [\delta, \frac{1}{\delta}]$ , for some  $\delta$  with  $0 < \delta < 1$ . This follows from a standard fixed-point argument; see, for example, [5, Section 3.5.2].

**Remark 3.2.** In [10], Faria *et al.* extend the results of [6] to the case when the initial potential density  $\nu_0$  is in  $L^1$ . The question of whether this can be done for the fully compressible semi-geostrophic equations remains open.

### 3.2 Existence and uniqueness for the energy minimisation

Assume that

- $\Omega$  and  $\Lambda$  are bounded open domains in  $\mathbb{R}^3$ ,
  - $\Omega$  is convex ;  $\overline{\Lambda} \subset \mathbb{R}^2 \times \left[ \delta, \frac{1}{\delta} \right]$  for some  $\delta$  with  $0 < \delta < 1$ ,
- (3.20)

where the assumption on the vertical coordinate of  $\Lambda$  is justified by (2.1)(vi). Cullen and Maroofi proved in [8, Theorem 4.1] that, given  $\nu \in P_{ac}(\Lambda)$ , there exists a unique  $\sigma \in P_{ac}(\Omega)$  that minimises the energy  $E_\nu(\cdot)$ :

**Theorem 3.3.** *Assume that  $\Omega$  and  $\Lambda$  satisfy (3.20). Let  $\nu \in P_{ac}(\Lambda)$  and assume that  $\phi$  satisfies (2.1)(ix). Then there exists a unique minimiser  $\sigma$  of  $E_\nu(\cdot)$ , over  $P_{ac}(\Omega)$ . This minimiser is given by (3.11).*

By fixing this minimising  $\sigma$ , we can complete the energy minimisation by considering (3.12) as an optimal transport problem. Indeed, given two probability densities  $\sigma \in P_{ac}(\Omega)$  and  $\nu \in P_{ac}(\Lambda)$ , consider the following optimal transport problem:

$$\inf_{\overline{\mathbf{T}} \# \sigma = \nu} I_\sigma[\overline{\mathbf{T}}], \quad \text{with } I_\sigma[\overline{\mathbf{T}}] := \int_{\Omega} c(\mathbf{x}, \overline{\mathbf{T}}(\mathbf{x})) \sigma(\mathbf{x}) d\mathbf{x}, \quad (3.21)$$

where  $c(\mathbf{x}, \mathbf{y})$ , given by (3.13), is the cost of transporting one unit of mass from  $\mathbf{x}$  to  $\mathbf{y}$ .

The Kantorovich relaxation of (3.21) amounts to finding  $\gamma \in \Gamma(\sigma, \nu)$  that minimises

$$\overline{I}[\gamma] := \int_{\Omega \times \Lambda} c(\mathbf{x}, \mathbf{y}) d\gamma(\mathbf{x}, \mathbf{y}), \quad (3.22)$$

with  $\Gamma(\sigma, \nu)$  given in Definition 2.2. Note that every transport map  $\overline{\mathbf{T}}$  generates a transport plan  $\gamma_{\overline{\mathbf{T}}} \in \Gamma(\sigma, \nu)$  defined by

$$\gamma_{\overline{\mathbf{T}}} := (\text{id}, \overline{\mathbf{T}}) \# \sigma,$$

such that  $I_\sigma[\overline{\mathbf{T}}] = \overline{I}[\gamma_{\overline{\mathbf{T}}}]$ .

Now consider the Kantorovich dual problem corresponding to (3.21) (see, for example, [14, Chapter 5]):

$$\sup_{(f, g) \in \text{Lip}_c} J_{(\sigma, \nu)}(f, g), \quad \text{with } J_{(\sigma, \nu)}(f, g) := \int_{\Omega} f(\mathbf{x}) \sigma(\mathbf{x}) d\mathbf{x} + \int_{\Lambda} g(\mathbf{y}) \nu(\mathbf{y}) d\mathbf{y}, \quad (3.23)$$

where

$$Lip_c := \{(f, g) : f \in W^{1,\infty}(\Omega), g \in W^{1,\infty}(\Lambda), f(\mathbf{x}) + g(\mathbf{y}) \leq c(\mathbf{x}, \mathbf{y}) \text{ for all } (\mathbf{x}, \mathbf{y}) \in \Omega \times \Lambda\}.$$

The solution of the dual problem is a crucial ingredient of proving the main result of this section, Theorem 3.5 below.

Let us discuss the dual formulation. We start by defining two operators that are standard in the theory of optimal transport.

**Definition 3.1.** Let  $\Omega$  and  $\Lambda$  be bounded open sets in  $\mathbb{R}^3$ , and let  $c(\mathbf{x}, \mathbf{y})$  be given by (3.13). Assume that  $f$  is a function from  $\overline{\Omega}$  into  $\mathbb{R}$  and  $g$  is a function from  $\overline{\Lambda}$  into  $\mathbb{R}$ . Then the  $c$ -transform of  $f$  on  $\Omega$  is defined as

$$f^c(\mathbf{y}) := \inf_{\mathbf{x} \in \overline{\Omega}} \{c(\mathbf{x}, \mathbf{y}) - f(\mathbf{x})\}, \quad \mathbf{y} \in \mathbb{R}^3 \setminus \{\mathbf{y} : y_3 = 0\}, \quad (3.24)$$

and the  $c$ -transform of  $g$  on  $\Lambda$  is defined as

$$g_c(\mathbf{x}) := \inf_{\mathbf{y} \in \overline{\Lambda}} \{c(\mathbf{x}, \mathbf{y}) - g(\mathbf{y})\}, \quad \mathbf{x} \in \mathbb{R}^3. \quad (3.25)$$

Cullen and Maroofi prove and utilise various useful properties of these  $c$ -transforms to show the existence of solutions of the Kantorovich dual problem. We summarise these results in the following (see [8, Lemma 3.1, Theorem 3.2] for proof):

**Theorem 3.4.** Assume  $\Omega$  and  $\Lambda$  satisfy (3.20). Let  $\sigma \in P_{ac}(\Omega)$  and  $\nu \in P_{ac}(\Lambda)$ . Assume that  $c(\mathbf{x}, \mathbf{y})$  is given by (3.13) and that  $\phi$  satisfies (2.1)(ix). Then the problem (3.23) of maximising  $J_{(\sigma, \nu)}$  over  $Lip_c$  has a solution  $(f_0, g_0)$  such that  $(f_0)^c = g_0$ ,  $(g_0)_c = f_0$  and

(i)

$$\|f_0\|_{W^{1,\infty}(\Omega)} + \|g_0\|_{W^{1,\infty}(\Lambda)} \leq C = C(\Omega, \Lambda, c(\cdot, \cdot));$$

(ii) for  $\nu$ -a.e.  $\mathbf{y} \in \Lambda$ , the infimum in (3.24) is attained at a unique point  $\overline{\mathbf{x}} = \overline{\mathbf{x}}(\mathbf{y})$  with

$$\nabla g_0(\mathbf{y}) = \nabla_{\mathbf{y}} c(\overline{\mathbf{x}}, \mathbf{y}), \quad (3.26)$$

and for  $\sigma$ -a.e.  $\mathbf{x} \in \Omega$ , the infimum in (3.25) is attained at a unique point  $\overline{\mathbf{y}} = \overline{\mathbf{y}}(\mathbf{x})$  with

$$\nabla f_0(\mathbf{x}) = \nabla_{\mathbf{x}} c(\mathbf{x}, \overline{\mathbf{y}}); \quad (3.27)$$

(iii)  $g_0$  is semi-concave in  $\Lambda$ ,  $f_0$  is semi-concave in  $\Omega$  and the constant of semi-concavity in both cases depends only on  $\Omega$ ,  $\Lambda$  and  $c(\cdot, \cdot)$ .

Finally, we state the main existence result for the optimal transport problem, see [8, Theorem 3.3] for a proof.

**Theorem 3.5.** Assume that  $\Omega$  and  $\Lambda$  satisfy (3.20). Let  $\sigma \in P_{ac}(\Omega)$  and  $\nu \in P_{ac}(\Lambda)$ . Assume that  $\phi$  satisfies (2.1)(ix). Then there exist maps  $\mathbf{T} : \Omega \rightarrow \Lambda$  and  $\mathbf{S} : \Lambda \rightarrow \Omega$ , unique  $\sigma$ -a.e. and  $\nu$ -a.e. respectively, such that

(i)  $\mathbf{T}$  is optimal in the transport of  $\sigma$  to  $\nu$  with cost  $c(\mathbf{x}, \mathbf{y})$ ,

(ii)  $\mathbf{S}$  is optimal in the transport of  $\nu$  to  $\sigma$  with cost  $\tilde{c}(\mathbf{y}, \mathbf{x}) = c(\mathbf{x}, \mathbf{y})$ ,

(iii)  $\mathbf{S}$  and  $\mathbf{T}$  are inverses, i.e.  $\mathbf{S} \circ \mathbf{T}(\mathbf{x}) = \mathbf{x}$  for  $\sigma$ -a.e.  $\mathbf{x}$  and  $\mathbf{T} \circ \mathbf{S}(\mathbf{y}) = \mathbf{y}$  for  $\nu$ -a.e.  $\mathbf{y}$ ,



(iv)  $\gamma_0 = (\mathbf{id}, \mathbf{T}) \# \sigma$  is a minimiser of the relaxed optimal transport problem (3.22),

(v) For  $J_{(\sigma, \nu)}(f, g)$  defined by (3.23), the following equality holds:

$$\sup_{(f, g) \in Lip_c} J_{(\sigma, \nu)}(f, g) = \inf_{\gamma \in \Gamma(\sigma, \nu)} \bar{I}[\gamma] = \inf_{\bar{\mathbf{T}} \# \sigma = \nu} I_\sigma[\bar{\mathbf{T}}].$$

## 4 The main existence result in dual space

### 4.1 Statement

We give the statement of the existence result for the equations (3.14)-(3.19) in dual variables. This theorem is given in [8, Theorem 5.5], where a proof is obtained by an approximation argument.

**Theorem 4.1.** *Let  $1 < r < \infty$  and  $\nu_0 \in L^r(\Lambda_0)$  be an initial potential density with support in  $\Lambda_0$ , where  $\Lambda_0$  is a bounded open set in  $\mathbb{R}^3$  with  $\bar{\Lambda}_0 \subset \mathbb{R}^2 \times [\tilde{\delta}, 1/\tilde{\delta}]$  for some  $0 < \tilde{\delta} < 1$ . Let  $\Omega$  be an open bounded convex set in  $\mathbb{R}^3$ . Assume that  $c(\cdot, \cdot)$  is given by (3.13) and that  $\phi$  satisfies (2.1)(ix). Then the system of semi-geostrophic equations in dual variables (3.14)-(3.19) has a stable weak solution  $(\sigma, \mathbf{T})$  such that, with  $\nu(t, \cdot) = \mathbf{T}(t, \cdot) \# \sigma(t, \cdot)$  and  $\mathbf{w}$  as in (3.15),*

$$(i) \quad \nu(\cdot, \cdot) \in L^r((0, \tau) \times \Lambda), \quad \|\nu(t, \cdot)\|_{L^r(\Lambda)} \leq \|\nu_0(\cdot)\|_{L^r(\Lambda)}, \quad \forall t \in [0, \tau],$$

$$(ii) \quad \sigma(t, \cdot) \in W^{1, \infty}(\Omega), \quad \|\sigma(t, \cdot)\|_{W^{1, \infty}(\Omega)} \leq C = C(\Omega, \Lambda, c(\cdot, \cdot), \kappa, K_1), \quad \forall t \in [0, \tau],$$

$$(iii) \quad \|\mathbf{w}(t, \cdot)\|_{L^\infty(\Lambda)} \leq C = C(\Omega, \Lambda, f_{cor}), \quad \forall t \in [0, \tau],$$

where  $\Lambda$  is a bounded open domain in  $\mathbb{R}^3$  containing  $\text{supp}(\nu)$ , such that  $\bar{\Lambda} \subset \mathbb{R}^2 \times [\delta, \frac{1}{\delta}]$  for some  $0 < \delta < 1$ .

### 4.2 Alternative proof of Theorem 4.1

Ambrosio and Gangbo in [2] proved a general result on the solution of Hamiltonian ODEs that may be applied to the dual space existence problem. We briefly summarise the work in [2] and then describe how it can be applied to the compressible semi-geostrophic equations we have been considering.

**Remark 4.2.** In what follows we deal with concave rather than convex functions. Hence we replace all definitions in [2] regarding subdifferentiability and  $(\lambda)$ -convexity with the following definitions regarding superdifferentiability and  $(\lambda)$ -concavity. This replacement does not affect the results of [2].

**Definition 4.1.** Let  $H : P_{ac}^2(\mathbb{R}^3) \rightarrow (-\infty, +\infty]$  be a proper, upper semicontinuous function and let  $\nu \in D(H)$ . We say that  $\mathbf{v} \in L^2(\nu; \mathbb{R}^3)$  belongs to the Fréchet superdifferential  $\partial H(\nu)$  if

$$H(\nu_h) \leq H(\nu) + \int_{\mathbb{R}^3} \mathbf{v}(\mathbf{y}) \cdot (\mathbf{R}_\nu^{\nu_h}(\mathbf{y}) - \mathbf{y}) \nu(\mathbf{y}) d\mathbf{y} + o(W_2(\nu, \nu_h))$$

as  $\nu_h \rightarrow \nu$ . We denote by  $\partial_0 H(\nu)$  the element of  $\partial H(\nu)$  of minimal  $L^2(\nu; \mathbb{R}^3)$ -norm. Note that, by the minimality of its norm,  $\partial_0 H(\nu)$  belongs to  $\partial H(\nu) \cap T_\nu P_{ac}^2(\mathbb{R}^3)$ .

In the following lemma we state a continuity property of optimal plans or maps.

**Lemma 4.3.** *Assume that  $\{\nu_n\}_{n=1}^\infty, \{\mu_n\}_{n=1}^\infty$  are bounded sequences in  $P_{ac}^2(\mathbb{R}^3)$  narrowly converging to  $\nu$  and  $\mu$  respectively. Assume that  $\Gamma_0(\nu, \mu)$  contains a unique plan  $\gamma_0$  induced by the optimal map  $\mathbf{R}_\nu^\mu : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  (see Remark 2.1). Then*

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^3} g(\mathbf{y}, \mathbf{R}_{\nu_n}^{\mu_n}(\mathbf{y})) \nu_n(\mathbf{y}) d\mathbf{y} = \int_{\mathbb{R}^3} g(\mathbf{y}, \mathbf{R}_\nu^\mu(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y},$$

where  $\mathbf{R}_{\nu_n}^{\mu_n}$  is optimal in the transport of  $\nu_n$  to  $\mu_n$ , and for any continuous function  $g : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$  satisfying

$$\lim_{|(\mathbf{y}, \mathbf{x})| \rightarrow +\infty} \frac{|g|(\mathbf{y}, \mathbf{x})}{|\mathbf{y}|^2 + |\mathbf{x}|^2} = 0.$$

Assume furthermore that there exists a closed ball  $B_r$ , of finite radius  $r$ , containing the supports of  $\mu_n$  and  $\mu$ . Then there exist Lipschitz, convex functions  $u_n, u : \mathbb{R}^3 \rightarrow \mathbb{R} \cup \{+\infty\}$  such that  $\nabla u_n = \mathbf{R}_{\nu_n}^{\mu_n} \nu_n - \text{a.e. in } \mathbb{R}^3$  and  $\nabla u = \mathbf{R}_\nu^\mu \nu - \text{a.e. in } \mathbb{R}^3$ . In addition, there exists a subsequence  $\{n_k\}_{k=1}^\infty$  of integers such that

$$\nabla u_{n_k} \rightarrow \nabla u \quad \text{a.e. in } \mathbb{R}^3. \quad (4.1)$$

*Proof.* See [3, Proposition 7.1.3] for proof.  $\square$

**Definition 4.2.** Let  $H : P_{ac}^2(\mathbb{R}^3) \rightarrow (-\infty, +\infty]$  be proper and let  $\lambda \in \mathbb{R}$ . Let  $\pi_1 : \mathbb{R}^3 \times \mathbb{R}^3 : (\mathbf{x}, \mathbf{y}) \rightarrow \mathbf{x}$  and  $\pi_2 : \mathbb{R}^3 \times \mathbb{R}^3 : (\mathbf{x}, \mathbf{y}) \rightarrow \mathbf{y}$  be the first and second projections of  $\mathbb{R}^3 \times \mathbb{R}^3$  onto  $\mathbb{R}^3$ . We say that  $H$  is  $\lambda$ -concave if for every  $\nu_1, \nu_2 \in P_{ac}^2(\mathbb{R}^3)$  and every optimal transport plan  $\gamma \in \Gamma_0(\nu_1, \nu_2)$  we have

$$H(\nu_{(t)}) \geq (1-t)H(\nu_1) + tH(\nu_2) - \frac{\lambda}{2}t(1-t)W_2^2(\nu_1, \nu_2)$$

for all  $t \in [0, 1]$ , where  $\nu_{(t)} = ((1-t)\pi_1 + t\pi_2)\#\gamma$ .

**Proposition 4.4.** Let  $H : P_{ac}^2(\mathbb{R}^3) \rightarrow (-\infty, +\infty]$  be upper semicontinuous and  $\lambda$ -concave for some  $\lambda \in \mathbb{R}$  and let  $\nu \in D(H)$ . Then, the following condition is equivalent to  $\mathbf{v} \in \partial H(\nu)$ :

$$H(\nu_h) \leq H(\nu) + \int_{\mathbb{R}^3} \mathbf{v}(\mathbf{y}) \cdot (\mathbf{R}_\nu^{\nu_h}(\mathbf{y}) - \mathbf{y}) \nu(\mathbf{y}) d\mathbf{y} + \frac{\lambda}{2}W_2^2(\nu, \nu_h)$$

for all  $\nu_h \in P_{ac}^2(\mathbb{R}^3)$ .

*Proof.* See [2, Proposition 4.2] for proof.  $\square$

Following [2], we define Hamiltonian ODE's as follows:

**Definition 4.3.** Let  $H : P_{ac}^2(\mathbb{R}^3) \rightarrow (-\infty, +\infty]$  be a proper, upper semicontinuous function. Define the linear transformation  $\tilde{J} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  by

$$\tilde{J}(v_1(\mathbf{y}), v_2(\mathbf{y}), v_3(\mathbf{y})) = f_{cor}^{-1}y_3(-v_2(\mathbf{y}), v_1(\mathbf{y}), 0), \quad (4.2)$$

for all  $\mathbf{v}(\mathbf{y}) \in \mathbb{R}^3$ . We say that an absolutely continuous curve  $\nu_{(t)} : [0, \tau] \rightarrow D(H)$  is a Hamiltonian ODE relative to  $H$ , starting from  $\nu_0 \in P_{ac}^2(\mathbb{R}^3)$ , if there exist  $\mathbf{v}_{(t)} \in L^2(\nu_{(t)}; \mathbb{R}^3)$  with  $\|\mathbf{v}_{(t)}\|_{L^2(\nu_{(t)})} \in L^1(0, \tau)$ , such that

$$\begin{cases} \frac{d}{dt}\nu_{(t)} + \nabla \cdot (\tilde{J}\mathbf{v}_{(t)}\nu_{(t)}) = 0, & \nu_{(0)} = \nu_0, & t \in (0, \tau) \\ \mathbf{v}_{(t)} \in T_{\nu_{(t)}}P_{ac}^2(\mathbb{R}^3) \cap \partial H(\nu_{(t)}) & \text{for a.e. } t. \end{cases} \quad (4.3)$$

The main result of [2] concerns Hamiltonians  $H$  satisfying the following properties:

(H1) *There exist constants  $C_0 \in (0, +\infty)$ ,  $R_0 \in (0, +\infty]$  such that for all  $\nu \in P_{ac}^2(\mathbb{R}^3)$  with  $W_2(\nu, \nu_0) < R_0$  we have  $\nu \in D(H)$ ,  $\partial H(\nu) \neq \emptyset$  and  $\mathbf{v} = \partial_0 H(\nu)$  satisfies  $|\mathbf{v}(\mathbf{z})| \leq C_0(1 + |\mathbf{z}|)$  for  $\nu$ -a.e.  $\mathbf{z} \in \mathbb{R}^3$ .*

(H2) *If  $\nu, \nu_n \in P_{ac}^2(\mathbb{R}^3)$ ,  $\sup_n W_2(\nu_n, \nu_0) < R_0$  and  $\nu_n \rightarrow \nu$  narrowly, then there exists a subsequence  $n(k)$  and functions  $\mathbf{v}_k, \mathbf{v}$  such that  $\mathbf{v}_k = \partial_0 H(\nu_{n(k)})$   $\nu_{n(k)}$ -a.e.,  $\mathbf{v} = \partial_0 H(\nu)$   $\nu$ -a.e. and  $\mathbf{v}_k \rightarrow \mathbf{v}$  a.e. in  $\mathbb{R}^3$  as  $k \rightarrow +\infty$ .*

To ensure the constancy of  $H$  along the solutions of the Hamiltonian system we consider also:

(H3)  *$H : P_{ac}^2(\mathbb{R}^3) \rightarrow (-\infty, +\infty]$  is proper, upper semicontinuous and  $\lambda$ -concave for some  $\lambda \in \mathbb{R}$ .*

For Hamiltonians  $H$  as above, the following result holds (see [2, Theorem 6.6] for full details and proof):

**Theorem 4.5.** *Assume that (H1) and (H2) hold for  $H(\nu)$  and that  $\tau > 0$  satisfies*

$$C_0 \tau \sqrt{24(1 + e^{(25C_0^2+1)\tau}(1 + M_2(\nu_0)))} < R_0. \quad (4.4)$$

*Then there exists an absolutely continuous Hamiltonian flow  $\nu_{(t)} \in P_{ac}^2(\mathbb{R}^3)$ ,  $\nu_{(t)} : [0, \tau] \rightarrow D(H)$  starting from  $\nu_0 \in P_{ac}^2(\mathbb{R}^3)$ , satisfying (4.3), such that the velocity field  $\mathbf{v}_{(t)}$  coincides with  $\partial_0 H(\nu_{(t)})$  for a.e.  $t \in [0, \tau]$ . Furthermore, the function  $t \rightarrow \nu_{(t)}$  is Lipschitz continuous. Finally, there exists a function  $l(r)$  depending only on  $\tau$  and  $C_0$  such that*

$$\nu_0 \geq m_r \text{ a.e. on } B_r \text{ for all } r > 0 \implies \nu_{(t)} \geq m_{l(r)} \text{ a.e. on } B_r \text{ for all } r > 0 \quad (4.5)$$

and

$$\nu_0 \leq M_r \text{ a.e. on } B_r \text{ for all } r > 0 \implies \nu_{(t)} \leq M_{l(r)} \text{ a.e. on } B_r \text{ for all } r > 0. \quad (4.6)$$

*If in addition (H3) holds, then  $t \mapsto H(\nu_{(t)})$  is constant.*

We now apply Theorem 4.5 to our problem in order to obtain directly the existence result of Theorem 4.1.

Let  $\Lambda, \Omega$  be as in (3.20). For  $\nu \in P_{ac}(\Lambda)$ , define the Hamiltonian  $H$  by

$$H(\nu) = \inf_{\mu \in P_{ac}(\Omega)} \left\{ \mathcal{E}(\nu, \mu) + K_1 \int_{\Omega} (\mu(\mathbf{x}))^\kappa d\mathbf{x} \right\}, \quad (4.7)$$

where

$$\mathcal{E}(\nu, \mu) = \inf_{\bar{\mathbf{S}} \# \nu = \mu} \int_{\Lambda} \tilde{c}(\mathbf{y}, \bar{\mathbf{S}}(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y} \quad (4.8)$$

with  $\tilde{c}(\mathbf{y}, \mathbf{x}) = c(\mathbf{x}, \mathbf{y})$  defined by (3.13). We begin with the following:

**Proposition 4.6.** *Let  $\Omega$  and  $\Lambda$  satisfy (3.20). Let the Hamiltonian  $H(\nu)$  on  $P_{ac}(\Lambda)$  be defined by (4.7). Then  $H$  is superdifferentiable, upper semicontinuous and  $(-2)$ -concave.*

*Proof.* Given  $\nu \in P_{ac}(\Lambda)$ , denote by  $\sigma$  the minimiser in (4.7). The existence and uniqueness of this minimiser follows from Theorem 3.3. For any  $\nu_h \in P_{ac}(\Lambda)$  we have

$$\begin{aligned} H(\nu_h) &= \inf_{\mu \in P_{ac}(\Omega)} \left\{ \mathcal{E}(\nu_h, \mu) + K_1 \int_{\Omega} (\mu(\mathbf{x}))^\kappa d\mathbf{x} \right\} \\ &\leq \mathcal{E}(\nu_h, \sigma) + K_1 \int_{\Omega} (\sigma(\mathbf{x}))^\kappa d\mathbf{x}. \end{aligned} \quad (4.9)$$

First, recall that we can guarantee that there exists a unique optimal transport map  $\mathbf{R}_\nu^{\nu_h}$  from  $\nu$  to  $\nu_h$  with respect to the Wasserstein cost function  $d(\mathbf{y}, \mathbf{y}_h) = \frac{1}{2}|\mathbf{y} - \mathbf{y}_h|^2$ ; see Remark 2.1.

We consider now transport with respect to the cost function  $\tilde{c}(\mathbf{y}, \mathbf{x}) = c(\mathbf{x}, \mathbf{y})$  given by (3.13). Let  $\mathbf{S}_\nu^\sigma$  be the optimal map in the transport of  $\nu$  to  $\sigma$  and let  $\mathbf{S}_{\nu_h}^\sigma$  be the optimal map in the transport of  $\nu_h$  to  $\sigma$ . Therefore, we have

$$\inf_{\mathbf{S} \# \nu = \sigma} \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y} = \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_\nu^\sigma(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y}$$

and

$$\inf_{\mathbf{S} \# \nu_h = \sigma} \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}(\mathbf{y})) \nu_h(\mathbf{y}) d\mathbf{y} = \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_h}^\sigma(\mathbf{y})) \nu_h(\mathbf{y}) d\mathbf{y}.$$

The existence of  $\mathbf{S}_\nu^\sigma$  and  $\mathbf{S}_{\nu_h}^\sigma$  follows from Theorem 3.5. Note that, since  $(\mathbf{S}_\nu^\sigma \circ (\mathbf{R}_\nu^{\nu_h})^{-1}) \# \nu_h = \sigma$  and since  $\mathbf{S}_{\nu_h}^\sigma$  is optimal in the transport of  $\nu_h$  to  $\sigma$ , we have

$$\int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_h}^\sigma(\mathbf{y})) \nu_h(\mathbf{y}) d\mathbf{y} \leq \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_\nu^\sigma \circ (\mathbf{R}_\nu^{\nu_h})^{-1}(\mathbf{y})) \nu_h(\mathbf{y}) d\mathbf{y}.$$

It follows that

$$\begin{aligned} H(\nu_h) - H(\nu) &\leq \mathcal{E}(\nu_h, \sigma) + K_1 \int_{\Omega} (\sigma(\mathbf{x}))^\kappa d\mathbf{x} - \mathcal{E}(\nu, \sigma) - K_1 \int_{\Omega} (\sigma(\mathbf{x}))^\kappa d\mathbf{x} \\ &= \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_h}^\sigma(\mathbf{y})) \nu_h(\mathbf{y}) d\mathbf{y} - \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_\nu^\sigma(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y} \\ &\leq \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_\nu^\sigma \circ (\mathbf{R}_\nu^{\nu_h})^{-1}(\mathbf{y})) \nu_h(\mathbf{y}) d\mathbf{y} - \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_\nu^\sigma(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y} \\ &= \int_{\Lambda} \tilde{c}(\mathbf{R}_\nu^{\nu_h}(\mathbf{y}), \mathbf{S}_\nu^\sigma(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y} - \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_\nu^\sigma(\mathbf{y})) \nu(\mathbf{y}) d\mathbf{y} \\ &= \int_{\Lambda} \left[ \tilde{c}(\mathbf{R}_\nu^{\nu_h}(\mathbf{y}), \mathbf{S}_\nu^\sigma(\mathbf{y})) - \tilde{c}(\mathbf{y}, \mathbf{S}_\nu^\sigma(\mathbf{y})) \right] \nu(\mathbf{y}) d\mathbf{y}, \\ &= \int_{\Lambda} \nabla \tilde{c}(\mathbf{y}, \mathbf{S}_\nu^\sigma(\mathbf{y})) \cdot [\mathbf{R}_\nu^{\nu_h}(\mathbf{y}) - \mathbf{y}] \nu(\mathbf{y}) d\mathbf{y} + o(W_2(\nu, \nu_h)). \end{aligned} \quad (4.10)$$

where we have used (2.5).

Hence, using Definition 4.1, we conclude that  $\nabla \tilde{c}(\mathbf{y}, \mathbf{S}_\nu^\sigma(\mathbf{y})) \in \partial H(\nu)$ . Thus,  $\partial H(\nu)$  is non-empty,  $H$  is superdifferentiable and we can use [14, Proposition 10.12] to conclude that  $H$  is semi-concave, i.e.

$$H \text{ is } (-2) - \text{concave}. \quad (4.11)$$

Also, from the narrow continuity of  $\mathcal{E}(\cdot, \cdot)$  (see [8, Theorem 3.4]) and the uniform convergence of  $\sigma$  as the minimiser of (4.7) (see [8, Lemma 4.3]), we have that

$$H \text{ is upper semicontinuous}. \quad (4.12)$$

From (4.11) and (4.12), we have that (H3) holds.  $\square$

The following proposition yields an alternative proof of the main result of [8]; Theorem 4.1.

**Proposition 4.7.** *Let  $1 < r < \infty$  and  $\nu_0 \in L^r(\Lambda_0)$  be an initial potential density with support in  $\Lambda_0$ , where  $\Lambda_0$  is a bounded open set in  $\mathbb{R}^3$  with  $\overline{\Lambda_0} \subset \mathbb{R}^2 \times [\tilde{\delta}, 1/\tilde{\delta}]$  for some  $0 < \tilde{\delta} < 1$ . Let  $\Omega$  and  $\Lambda$  satisfy (3.20) and let the Hamiltonian  $H$  be defined by (4.7). Then, there exists a Hamiltonian flow  $\nu(t) \in P_{ac}(\Lambda)$  and constant  $\tau > 0$  such that*

$$\frac{d}{dt}\nu(t) + \nabla \cdot (\tilde{J}(\mathbf{v}(t))\nu(t)) = 0, \quad \nu_{(0)} = \nu_0, \quad t \in (0, \tau)$$

where  $\tilde{J}(\mathbf{v}(t)) = \mathbf{w}$  a.e. in  $[0, \tau]$ .

*Proof.* We compute  $\partial_0 H(\nu)$  (as defined in Definition 4.1) explicitly to show that the conditions required to apply Theorem 4.5 hold. From the definition of  $\tilde{J}$  in (4.2), velocity fields transporting  $\nu$  will have vanishing components in the  $y_3$  direction so that we need only consider variations of  $\nu$  in the  $(y_1, y_2)$ -directions. Thus, to characterise the elements of  $\partial H(\nu)$ , we let  $\varphi \in C_c^\infty(\mathbb{R}^3)$  be arbitrary, but with no  $y_3$  dependence, and set

$$\mathbf{g}_s(\mathbf{y}) = ((g_s)_1(\mathbf{y}), (g_s)_2(\mathbf{y}), (g_s)_3(\mathbf{y})) = \mathbf{y} + s\nabla\varphi(\mathbf{y}).$$

Note that  $(g_s)_3(\mathbf{y}) = y_3$  and, for  $|s|$  sufficiently small,  $\mathbf{g}_s$  is the gradient of a convex function, since  $\mathbf{g}_s(\mathbf{y}) = \nabla(\frac{1}{2}\mathbf{y}^2 + s\varphi)$ . Define  $\nu_s = \mathbf{g}_s\#\nu$ . Denote by  $\sigma_s$  the minimiser in

$$H(\nu_s) = \inf_{\mu \in P_{ac}(\Omega)} \left\{ \mathcal{E}(\nu_s, \mu) + K_1 \int_{\Omega} (\mu(\mathbf{x}))^\kappa d\mathbf{x} \right\}.$$

The existence and uniqueness of the minimiser  $\sigma_s$  follows from Theorem 3.3. Let  $\xi \in \partial H(\nu)$ . Combining the  $(-2)$ -concavity of  $H$  and (4.12) with Proposition 4.4, we obtain

$$H(\nu_s) - H(\nu) - \int_{\Lambda} \xi(\mathbf{y}) \cdot (\mathbf{R}_{\nu_s}^{\nu_s}(\mathbf{y}) - \mathbf{y})\nu(\mathbf{y}) d\mathbf{y} + W_2^2(\nu, \nu_s) \leq 0. \quad (4.13)$$

Brenier's polar factorisation theorem states that any suitable mapping from  $\nu$  to  $\nu_s$  can be uniquely factorised as the composition of a measure-preserving mapping and the gradient of a convex function (see, for example, [13, Chapter 3]). Since, for  $|s|$  sufficiently small,  $\mathbf{g}_s$  is the gradient of a convex function, we conclude that

$$W_2^2(\nu, \nu_s) = \int_{\Lambda} |\mathbf{y} - \mathbf{R}_{\nu_s}^{\nu_s}(\mathbf{y})|^2 \nu(\mathbf{y}) d\mathbf{y} = \int_{\Lambda} |\mathbf{y} - \mathbf{g}_s(\mathbf{y})|^2 \nu(\mathbf{y}) d\mathbf{y} = s^2 \int_{\Lambda} |\nabla\varphi(\mathbf{y})|^2 \nu(\mathbf{y}) d\mathbf{y}$$

and

$$\int_{\Lambda} \xi(\mathbf{y}) \cdot (\mathbf{R}_{\nu_s}^{\nu_s}(\mathbf{y}) - \mathbf{y})\nu(\mathbf{y}) d\mathbf{y} = \int_{\Lambda} \xi(\mathbf{y}) \cdot (\mathbf{g}_s(\mathbf{y}) - \mathbf{y})\nu(\mathbf{y}) d\mathbf{y} = s \int_{\Lambda} \xi(\mathbf{y}) \cdot \nabla\varphi(\mathbf{y})\nu(\mathbf{y}) d\mathbf{y}.$$

Combining this with (4.13), we therefore obtain

$$\begin{aligned} -s \int_{\Lambda} \xi(\mathbf{y}) \cdot \nabla\varphi(\mathbf{y})\nu(\mathbf{y}) d\mathbf{y} + s^2 \int_{\Lambda} |\nabla\varphi(\mathbf{y})|^2 \nu(\mathbf{y}) d\mathbf{y} &\leq H(\nu) - H(\nu_s) \\ &\leq \mathcal{E}(\nu, \sigma_s) - \mathcal{E}(\nu_s, \sigma_s) \\ &= \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y}))\nu(\mathbf{y}) d\mathbf{y} - \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y}))\nu_s(\mathbf{y}) d\mathbf{y} \\ &\leq \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_s}^{\sigma_s} \circ \mathbf{g}_s(\mathbf{y}))\nu(\mathbf{y}) d\mathbf{y} - \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y}))\nu_s(\mathbf{y}) d\mathbf{y} \end{aligned}$$

$$= \int_{\Lambda} \tilde{c}(\mathbf{g}_s^{-1}(\mathbf{y}), \mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y})) \nu_s(\mathbf{y}) d\mathbf{y} - \int_{\Lambda} \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y})) \nu_s(\mathbf{y}) d\mathbf{y}, \quad (4.14)$$

since  $\mathbf{g}_s \# \nu = \nu_s$ . Here  $\mathbf{S}_{\nu_s}^{\sigma_s}$  denotes the optimal transport map from  $\nu$  to  $\sigma_s$  and  $\mathbf{S}_{\nu_s}^{\sigma_s}$  denotes the optimal transport map from  $\nu_s$  to  $\sigma_s$  with respect to the cost function  $\tilde{c}(\cdot, \cdot)$ . The existence of  $\mathbf{S}_{\nu_s}^{\sigma_s}$  and  $\mathbf{S}_{\nu_s}^{\sigma_s}$  follows from Theorem 3.5.

Note that

$$\mathbf{g}_s^{-1}(\mathbf{y}) = \mathbf{y} - s \nabla \varphi(\mathbf{y}) + \frac{s^2}{2} \nabla^2 \varphi(\mathbf{y}) \nabla \varphi(\mathbf{y}) + \epsilon(s, \mathbf{y}),$$

where  $\epsilon$  is a function such that  $|\epsilon(s, \mathbf{y})| \leq |s|^3 \|\varphi\|_{C^3(\mathbb{R}^3)}$ .

Combining this expression for  $\mathbf{g}_s^{-1}$  with (4.14) and using  $\frac{\partial}{\partial y_3} \varphi = 0$ , we conclude that

$$\begin{aligned} & -s \int_{\Lambda} \xi(\mathbf{y}) \cdot \nabla \varphi(\mathbf{y}) \nu(\mathbf{y}) d\mathbf{y} + s^2 \int_{\Lambda} |\nabla \varphi(\mathbf{y})|^2 \nu(\mathbf{y}) d\mathbf{y} \leq \int_{\Lambda} [\tilde{c}(\mathbf{g}_s^{-1}(\mathbf{y}), \mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y})) - \tilde{c}(\mathbf{y}, \mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y}))] \nu_s(\mathbf{y}) d\mathbf{y} \\ & = \int_{\Lambda} \left[ \frac{\frac{1}{2} f_{\text{cor}}^2 \left\{ |(g_s)_1^{-1}(\mathbf{y}) - (S_{\nu_s}^{\sigma_s})_1(\mathbf{y})|^2 + |(g_s)_2^{-1}(\mathbf{y}) - (S_{\nu_s}^{\sigma_s})_2(\mathbf{y})|^2 \right\} + \phi(\mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y}))}{y_3} \right. \\ & \quad \left. - \frac{\frac{1}{2} f_{\text{cor}}^2 \left\{ |y_1 - (S_{\nu_s}^{\sigma_s})_1(\mathbf{y})|^2 + |y_2 - (S_{\nu_s}^{\sigma_s})_2(\mathbf{y})|^2 \right\} + \phi(\mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y}))}{y_3} \right] \nu_s(\mathbf{y}) d\mathbf{y} \\ & = \int_{\Lambda} \frac{1}{y_3} \left[ \frac{1}{2} f_{\text{cor}}^2 \left\{ \left| y_1 - s \frac{\partial}{\partial y_1} \varphi(\mathbf{y}) - (S_{\nu_s}^{\sigma_s})_1(\mathbf{y}) \right|^2 + \left| y_2 - s \frac{\partial}{\partial y_2} \varphi(\mathbf{y}) - (S_{\nu_s}^{\sigma_s})_2(\mathbf{y}) \right|^2 \right\} \right. \\ & \quad \left. - \frac{1}{2} f_{\text{cor}}^2 \left\{ |y_1 - (S_{\nu_s}^{\sigma_s})_1(\mathbf{y})|^2 + |y_2 - (S_{\nu_s}^{\sigma_s})_2(\mathbf{y})|^2 \right\} \right] \nu_s(\mathbf{y}) d\mathbf{y} + o(s) \\ & = s f_{\text{cor}}^2 \int_{\Lambda} \left( \frac{\mathbf{S}_{\nu_s}^{\sigma_s}(\mathbf{y}) - \mathbf{y}}{y_3} \right) \cdot \nabla \varphi(\mathbf{y}) \nu_s(\mathbf{y}) d\mathbf{y} + o(s). \end{aligned}$$

Recall now that  $\nu_s \rightarrow \nu$  in  $P_{ac}(\Lambda)$  and  $\sigma_s \rightarrow \sigma$  in  $P_{ac}(\Omega)$  as  $s \rightarrow 0$ , hence Theorem 4.3 gives

$$-s \int_{\Lambda} \xi(\mathbf{y}) \cdot \nabla \varphi(\mathbf{y}) \nu(\mathbf{y}) d\mathbf{y} + s^2 \int_{\Lambda} |\nabla \varphi(\mathbf{y})|^2 \nu(\mathbf{y}) d\mathbf{y} \leq s f_{\text{cor}}^2 \int_{\Lambda} \left( \frac{\mathbf{S}_{\nu}^{\sigma}(\mathbf{y}) - \mathbf{y}}{y_3} \right) \cdot \nabla \varphi(\mathbf{y}) \nu(\mathbf{y}) d\mathbf{y} + o(s).$$

Dividing both sides first by  $s > 0$ , then by  $s < 0$  and letting  $|s| \rightarrow 0$  we obtain

$$- \int_{\Lambda} \xi(\mathbf{y}) \cdot \nabla \varphi(\mathbf{y}) \nu(\mathbf{y}) d\mathbf{y} = f_{\text{cor}}^2 \int_{\Lambda} \left( \frac{\mathbf{S}_{\nu}^{\sigma}(\mathbf{y}) - \mathbf{y}}{y_3} \right) \cdot \nabla \varphi(\mathbf{y}) \nu(\mathbf{y}) d\mathbf{y}.$$

Thus, we have that  $\tilde{J}(\pi_{\nu} \xi(\mathbf{y})) = \tilde{J} \left( f_{\text{cor}}^2 \left( \frac{\mathbf{S}_{\nu}^{\sigma}(\mathbf{y}) - \mathbf{y}}{y_3} \right) \right)$ , where  $\pi_{\nu} : L^2(\nu; \Lambda) \rightarrow T_{\nu} P_{ac}^2(\Lambda)$  denotes the canonical orthogonal projection, where the tangent space is defined in (2.3). The minimality of the norm of  $\partial_0 H$  then gives

$$\tilde{J}(\partial_0 H(\nu)) = \tilde{J} \left( f_{\text{cor}}^2 \left( \frac{\mathbf{S}_{\nu}^{\sigma}(\mathbf{y}) - \mathbf{y}}{y_3} \right) \right) = \mathbf{w}(\mathbf{y}), \quad (4.15)$$

where  $\mathbf{w}$  is defined as in (3.15).

We can now check directly that conditions (H1) and (H2) hold. Using Theorem 3.4 and Theorem 3.5, we may conclude from (4.15) that

$$\tilde{J}(\partial_0 H(\nu)) = \mathbf{w}(\mathbf{y}) = \tilde{J}(\nabla g_0(\mathbf{y})).$$

Condition (H1) then follows from the fact that, as a solution of the dual problem (3.23),  $g_0 \in W^{1,\infty}(\Lambda)$ . Condition (H2) follows from the stability of optimal maps (see, for example, [14, Corollary 5.23]). Hence we may apply the result of Theorem 4.5 to conclude that there exists a Hamiltonian flow  $\nu_{(t)}$  such that

$$\frac{d}{dt}\nu_{(t)} + \nabla \cdot (\tilde{J}(\mathbf{v}_{(t)})\nu_{(t)}) = 0, \quad \nu_{(0)} = \nu_0, \quad t \in (0, \tau)$$

where  $\tilde{J}(\mathbf{v}_{(t)}) = \tilde{J}(\partial_0 H(\nu_{(t)}))$  for *a.e.*  $t \in [0, \tau]$ . By (4.15), this then completes the proof that the dual space continuity equation (3.14), with velocity field defined as in (3.15), is satisfied. In addition, from (H3) and the definition of  $\tilde{J}$ , the energy associated with the flow is conserved.  $\square$

### Concluding remarks on the dual space result

The result of the previous section yields a proof of Theorem 4.1 alternative to the existing proof of [8]. Indeed, by the definition of  $\mathbf{w}$  in Proposition 4.7, we have that  $(\sigma, \mathbf{T})$  is a stable solution of (3.14)-(3.19), where  $\mathbf{T} = \mathbf{S}^{-1}$  (see Theorem 3.5). Theorem 4.1 (i) follows from (4.5), (4.6); Theorem 4.1 (ii) follows from the fact that  $\sigma$  is a minimiser of (4.7) (see [8, Theorem 4.2]); Theorem 4.1 (iii) follows from the definition of  $\mathbf{w}$  in terms of  $\nabla g_0$  and the fact that, as a solution of the dual problem (3.23),  $g_0 \in W^{1,\infty}(\Lambda)$ .

## 5 Lagrangian statement of the equations in physical space

We aim to prove the existence of a weak Lagrangian solution of the fully compressible semi-geostrophic system (3.1)-(3.6). We will show that, with additional regularity, a weak Lagrangian solution would determine a weak Eulerian solution.

We begin by rewriting the system (3.1)-(3.6) in a form that enables us to state the equations in Lagrangian form. These new equations (with prescribed initial conditions) are

$$D_t \mathbf{T}(t, \mathbf{x}) = f_{\text{cor}} \mathbf{e}_3 \times [\mathbf{T}(t, \mathbf{x}) - \mathbf{x}], \quad (5.1)$$

$$\partial_t \sigma(t, \mathbf{x}) + \nabla \cdot (\sigma(t, \mathbf{x}) \mathbf{u}(t, \mathbf{x})) = 0, \quad (5.2)$$

$$\nabla_{\mathbf{x}} c(\mathbf{x}, \mathbf{T}(t, \mathbf{x})) + \kappa K_1 \nabla((\sigma(t, \mathbf{x}))^{\kappa-1}) = 0, \quad (5.3)$$

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } [0, \tau) \times \partial\Omega, \quad (5.4)$$

$$\sigma(0, \mathbf{x}) = \sigma_0(\mathbf{x}), \quad \mathbf{T}(0, \mathbf{x}) = \mathbf{T}_0, \quad \mathbf{T}_0 \# \sigma_0 \in L^r(\Lambda_0), \quad \Lambda_0 \subset \mathbb{R}^3 \text{ compact}, \quad (5.5)$$

where  $r \in (1, \infty)$ ,  $c(\mathbf{x}, \mathbf{T}(t, \mathbf{x}))$  is defined in (3.13) and  $K_1$  is defined as in (3.9).

**Proposition 5.1.** *A solution of (5.1)-(5.5) determines a solution of the original system (3.1)-(3.6).*

*Proof.* Given a solution  $(\mathbf{u}, \mathbf{T}, \sigma)$  of (5.1)-(5.5), we set the function  $\mathbf{u}$  in (3.1)-(3.6) to be equal to the function  $\mathbf{u}$  in (5.1)-(5.5) and we define

$$\theta(t, \mathbf{x}) := T_3(t, \mathbf{x}), \quad \rho(t, \mathbf{x}) := \frac{\sigma(t, \mathbf{x})}{\theta(t, \mathbf{x})}, \quad \mathbf{u}^g(t, \mathbf{x}) := f_{\text{cor}} \mathbf{e}_3 \times [\mathbf{T}(t, \mathbf{x}) - \mathbf{x}]. \quad (5.6)$$

Then (3.2) follows from (5.6) and equation  $D_t T_3 = 0$  of (5.1). Using this together with (5.6) and the fact that (5.2) is satisfied, we see that (3.3) holds.

In order to show that (3.4) holds, we first rearrange (3.5) to obtain  $p = R^\kappa p_{\text{ref}}^{1-\kappa} (\rho\theta)^\kappa$ . Thus,

$$\nabla p = R^\kappa p_{\text{ref}}^{1-\kappa} \kappa (\rho\theta)^{\kappa-1} \nabla(\rho\theta). \quad (5.7)$$

Then, substituting (5.6) into (5.3) gives

$$f_{\text{cor}} \mathbf{e}_3 \times \mathbf{u}^g + \nabla \phi + \theta \kappa K_1 \nabla(\rho\theta)^{\kappa-1} = 0.$$

Recalling that  $K_1 = c_v \left( \frac{R}{p_{\text{ref}}} \right)^{\kappa-1}$ , this becomes

$$f_{\text{cor}} \mathbf{e}_3 \times \mathbf{u}^g + \nabla \phi + \theta \kappa c_v \left( \frac{R}{p_{\text{ref}}} \right)^{\kappa-1} \nabla(\rho\theta)^{\kappa-1} = 0,$$

and therefore

$$f_{\text{cor}} \mathbf{e}_3 \times \mathbf{u}^g + \nabla \phi + \frac{1}{\rho} \kappa (\kappa - 1) c_v \left( \frac{R}{p_{\text{ref}}} \right)^{\kappa-1} (\rho\theta)^{\kappa-1} \nabla(\rho\theta) = 0.$$

Then, using (2.1)(xiv), (2.1)(xv) and (5.7) we obtain

$$f_{\text{cor}} \mathbf{e}_3 \times \mathbf{u}^g + \nabla \phi + \frac{1}{\rho} \nabla p = 0$$

and thus conclude that (3.4) is satisfied.

Finally, we obtain (3.1) using (3.4) together with the first two components of (5.1) and the definition of  $\mathbf{u}^g$  in (5.6). Hence, we obtain a solution of the original system of semi-geostrophic equations.  $\square$

We define a weak solution of (5.1)-(5.5) as follows:

**Definition 5.1.** Let  $\Omega$  be an open bounded convex set in  $\mathbb{R}^3$ . Let  $\mathbf{u} : [0, \tau) \times \Omega \rightarrow \mathbb{R}^3$  satisfy  $\mathbf{u} \in L^1([0, \tau) \times \Omega)$  and let  $\mathbf{T} : [0, \tau) \times \Omega \rightarrow \mathbb{R}^3$  satisfy  $\mathbf{T} \in L^\infty([0, \tau) \times \Omega)$ . Also, let  $\sigma(t, \cdot) \in W^{1, \infty}(\Omega)$  for all  $t \in [0, \tau]$ . Then  $(\mathbf{u}, \mathbf{T}, \sigma)$  is a weak (Eulerian) solution of (5.1)-(5.5) if

$$\begin{aligned} \int_{[0, \tau) \times \Omega} \{ \mathbf{T}(t, \mathbf{x}) \cdot [\partial_t \varphi(t, \mathbf{x}) + (\mathbf{u}(t, \mathbf{x}) \cdot \nabla) \varphi(t, \mathbf{x})] + f_{\text{cor}} \mathbf{e}_3 \times [\mathbf{T}(t, \mathbf{x}) - \mathbf{x}] \cdot \varphi(t, \mathbf{x}) \} \sigma(t, \mathbf{x}) dt d\mathbf{x} \\ + \int_{\Omega} \mathbf{T}_0(\mathbf{x}) \cdot \varphi(0, \mathbf{x}) \sigma_0(\mathbf{x}) d\mathbf{x} = 0, \end{aligned} \quad (5.8)$$

for any  $\varphi \in C_c^1([0, \tau) \times \Omega)$ , and

$$\int_{[0, \tau) \times \Omega} \{ \partial_t \psi(t, \mathbf{x}) + (\mathbf{u}(t, \mathbf{x}) \cdot \nabla) \psi(t, \mathbf{x}) \} \sigma(t, \mathbf{x}) dt d\mathbf{x} + \int_{\Omega} \psi(0, \mathbf{x}) \sigma_0(\mathbf{x}) d\mathbf{x} = 0, \quad (5.9)$$

for any  $\psi \in C_c^1([0, \tau) \times \Omega)$ .



Given a solution  $\mathbf{T}(t, \mathbf{x})$  and  $\mathbf{x}(t, \mathbf{y}) = \mathbf{S}(t, \mathbf{y})$  of the system (3.14)-(3.19) in dual coordinates, formally we have that  $(\mathbf{u}, \mathbf{T}, \sigma)$  satisfy (5.1)-(5.5). However, due to the low regularity of  $\mathbf{T}(t, \mathbf{x})$  and thus  $\mathbf{S}(t, \mathbf{y})$  obtained as a weak solution of (3.14)-(3.19), this argument is not rigorous. Indeed, the regularity obtained in [8] may not be enough to provide a solution of the Eulerian problem.

**Remark 5.2.** In a recent development, [?], the existence of weak Eulerian solutions of the incompressible semi-geostrophic equations on the 2-dimensional torus has been proved. Whilst the authors expect that this result could be extended to three dimensions, the question of whether it is possible to prove that weak Eulerian solutions exist for the fully compressible semi-geostrophic equations remains open.

Instead, we seek to find weak Lagrangian solutions of the problem (5.1)-(5.5). Here we define such solutions and then we use the methods of [6] to prove their existence. We define the Lagrangian flow map

$$\mathbf{F} : [0, \tau) \times \Omega \rightarrow \Omega$$

corresponding to the velocity  $\mathbf{u}$ . Note that  $\mathbf{F}$  maps  $\Omega$  to itself so that the boundary conditions of the problem are respected. Then, we can rewrite the system (5.1)-(5.5) in terms of this Lagrangian flow  $\mathbf{F}$  and define the corresponding weak solution  $(\mathbf{F}, \mathbf{T}, \sigma)$ .

**Definition 5.2.** Let  $\Omega$  be an open bounded convex set in  $\mathbb{R}^3$  and let  $\tau > 0$ . Let  $\mathbf{T} \in L^\infty([0, \tau) \times \Omega)$  and let  $\sigma(t, \cdot) \in W^{1, \infty}(\Omega)$  for all  $t \in [0, \tau]$ . For  $1 \leq q < \infty$ , let  $\mathbf{F} : [0, \tau) \times \Omega \rightarrow \Omega$  be a Borel map, satisfying

$$\mathbf{F} \in C([0, \tau); L^q(\Omega; \mathbb{R}^3)). \quad (5.10)$$

Then  $(\mathbf{F}, \mathbf{T}, \sigma)$  is called a weak Lagrangian solution of (5.1)-(5.5) in  $[0, \tau) \times \Omega$  if:

(i) For  $\sigma_0$  - a.e.  $\mathbf{x} \in \Omega$ ,

$$\mathbf{F}(0, \mathbf{x}) = \mathbf{x}, \quad \mathbf{T}(0, \mathbf{x}) = \mathbf{T}_0(\mathbf{x}).$$

(ii) For any  $t > 0$ , the mapping

$$\mathbf{F}_{(t)} = \mathbf{F}(t, \cdot) : [0, \tau) \times \Omega \rightarrow \Omega$$

pushes forward the probability measure  $\sigma_0(\cdot)$  to  $\sigma(t, \cdot)$ , i.e.

$$\mathbf{F}_{(t)} \# \sigma_0(\cdot) = \sigma(t, \cdot). \quad (5.11)$$

(iii) There exists a Borel map

$$\mathbf{F}^* : [0, \tau) \times \Omega \rightarrow \Omega$$

such that, for every  $t \in (0, \tau)$ , the map

$$\mathbf{F}_{(t)}^* = \mathbf{F}^*(t, \cdot) : \Omega \rightarrow \Omega$$

satisfies

$$\mathbf{F}_{(t)}^* \circ \mathbf{F}_{(t)}(\mathbf{x}) = \mathbf{x} \text{ and } \mathbf{F}_{(t)} \circ \mathbf{F}_{(t)}^*(\mathbf{x}) = \mathbf{x},$$

for  $\sigma$  - a.e.  $\mathbf{x} \in \Omega$ , and

$$\mathbf{F}_{(t)}^* \# \sigma(t, \cdot) = \sigma_0(\cdot).$$

(iv) The function

$$\mathbf{Z}(t, \mathbf{x}) := \mathbf{T}(t, \mathbf{F}_{(t)}(\mathbf{x})) \quad (5.12)$$

is a weak solution of

$$\begin{aligned} \partial_t \mathbf{Z}(t, \mathbf{x}) &= f_{cor} \mathbf{e}_3 \times [\mathbf{Z}(t, \mathbf{x}) - \mathbf{F}(t, \mathbf{x})] \quad \text{in } [0, \tau) \times \Omega, \\ \mathbf{Z}(0, \mathbf{x}) &= \mathbf{T}_0(\mathbf{x}) \quad \text{in } \Omega, \end{aligned} \quad (5.13)$$

in the following sense:

for any  $\varphi \in C_c^1([0, \tau) \times \Omega; \mathbb{R}^3)$ ,

$$\begin{aligned} \int_{[0, \tau) \times \Omega} \{ \mathbf{Z}(t, \mathbf{x}) \cdot \partial_t \varphi(t, \mathbf{x}) + f_{cor} \mathbf{e}_3 \times [\mathbf{Z}(t, \mathbf{x}) - \mathbf{F}(t, \mathbf{x})] \cdot \varphi(t, \mathbf{x}) \} \sigma_0(\mathbf{x}) dt d\mathbf{x} \\ + \int_{\Omega} [\mathbf{T}_0(\mathbf{x}) \cdot \varphi(0, \mathbf{x})] \sigma_0(\mathbf{x}) d\mathbf{x} = 0. \end{aligned} \quad (5.14)$$

**Remark 5.3.** Let us comment on Definition 5.2:

- Continuity in time of  $\mathbf{F}$  considered as a map on  $[0, \tau)$  with values in  $L^q(\Omega)$  as required in (5.10), combined with the initial condition for  $\mathbf{F}$  in (i), implies that

$$\lim_{t \rightarrow 0^+} \|\mathbf{F}_{(t)} - \mathbf{id}\|_{L^q(\Omega)} = 0. \quad (5.15)$$

Furthermore, the continuity property (5.10) may be interpreted as continuity of particle paths in physical space.

- Property (ii) is the Lagrangian form of the mass conservation principle in equation (5.2) with boundary condition (5.4).
- Equation (5.13) is the Lagrangian form of (5.1).
- Equation (5.14) is derived by multiplying (5.13) by  $\sigma_0(\mathbf{x})$  as well as the test function and then integrating by parts, as in [6, Definition 3.4]. The reason for this choice will become clear in the proof of Proposition 5.4.
- We have omitted equation (5.3) since this holds as a result of the energy minimisation; see [8, Theorem 4.2].

In order to justify Definition 5.2, we must show that a weak Lagrangian solution corresponds to a weak (Eulerian) solution of (5.1)-(5.5), as defined by Definition 5.1. Indeed, we prove that, with additional regularity property  $\partial_t \mathbf{F} \in L^\infty([0, \tau) \times \Omega)$ , a weak Lagrangian solution  $(\mathbf{F}, \mathbf{T}, \sigma)$  as defined in Definition 5.2 determines a weak (Eulerian) solution of (5.1)-(5.5) and, furthermore, that a smooth Lagrangian solution determines a classical solution of (5.1)-(5.5). This is the content of the following result:

**Proposition 5.4.** *Let  $\Omega$  be an open bounded convex set in  $\mathbb{R}^3$  and let  $\tau > 0$ . Let  $(\mathbf{F}, \mathbf{T}, \sigma)$  be a weak Lagrangian solution of (5.1)-(5.5) in  $[0, \tau) \times \Omega$ .*

(i) *If  $\partial_t \mathbf{F} \in L^\infty([0, \tau) \times \Omega; \mathbb{R}^3)$ , then the function*

$$\mathbf{u}(t, \mathbf{x}) := (\partial_t \mathbf{F})(t, \mathbf{F}_{(t)}^*(\mathbf{x})) \quad (5.16)$$

*satisfies  $\mathbf{u} \in L^\infty([0, \tau) \times \Omega; \mathbb{R}^3)$  and  $(\mathbf{u}, \mathbf{T}, \sigma)$  is a weak Eulerian solution of (5.1)-(5.5) in  $[0, \tau) \times \Omega$  in the sense of Definition 5.1.*

(ii) If  $(\mathbf{F}, \mathbf{F}^*, \mathbf{T}) \in C^2([0, \tau] \times \overline{\Omega})$ , then the function (5.16) satisfies  $\mathbf{u} \in C^1([0, \tau] \times \overline{\Omega}; \mathbb{R}^3)$ , and  $(\mathbf{u}, \mathbf{T}, \sigma)$  is a classical solution of (5.1)-(5.5) in  $[0, \tau] \times \Omega$ .

*Proof.* Let us first prove (i). Since  $\mathbf{F}^*$  is a Borel map and, by our additional regularity assumption,  $\partial_t \mathbf{F} \in L^\infty([0, \tau] \times \Omega)$ , we know that the right-hand side of (5.16) is a bounded measurable function. Therefore,  $\mathbf{u} \in L^\infty([0, \tau] \times \Omega)$ . Now, in order to prove that  $(\mathbf{u}, \mathbf{T}, \sigma)$  is a weak Eulerian solution of (5.1)-(5.5), we must show that (5.8) and (5.9) hold. We begin with (5.9). Let  $\psi \in C_c^1([0, \tau] \times \overline{\Omega})$ , so that the support of  $\psi$  in  $t$  is a closed subset of  $[0, \tau]$ , and fix  $t \in (0, \tau)$ . Note that, since  $\sigma(t, \cdot) \in W^{1, \infty}(\Omega)$  for all  $t \in [0, \tau]$  (by Theorem 4.1 (ii)) we have that  $\sigma(t, \cdot) \in L^1([0, \tau] \times \Omega)$  and, since  $\mathbf{F}_{(t)} \# \sigma_0(\cdot) = \sigma(t, \cdot)$ , we can apply (2.5) to yield

$$\int_{\Omega} (\partial_t \psi)(t, \mathbf{F}_{(t)}(\mathbf{x})) \sigma_0(\mathbf{x}) d\mathbf{x} = \int_{\Omega} \partial_t \psi(t, \mathbf{X}) \sigma(t, \mathbf{X}) d\mathbf{X}.$$

Then, applying the chain rule, integrating both sides with respect to  $t$  and using our assumption that  $\partial_t \mathbf{F} \in L^\infty([0, \tau] \times \Omega)$ , we obtain

$$\begin{aligned} \int_{[0, \tau] \times \Omega} \{ \partial_t [\psi(t, \mathbf{F}_{(t)}(\mathbf{x}))] - \partial_t \mathbf{F}_{(t)}(\mathbf{x}) \cdot (\nabla \psi)(t, \mathbf{F}_{(t)}(\mathbf{x})) \} \sigma_0(\mathbf{x}) dt d\mathbf{x} \\ = \int_{[0, \tau] \times \Omega} \partial_t \psi(t, \mathbf{X}) \sigma(t, \mathbf{X}) dt d\mathbf{X}. \end{aligned}$$

Now, using (5.15) and the fact that  $\psi(\tau, \cdot) \equiv 0$  (by compact support), we get

$$\int_{[0, \tau] \times \Omega} \partial_t [\psi(t, \mathbf{F}_{(t)}(\mathbf{x}))] \sigma_0(\mathbf{x}) dt d\mathbf{x} = - \int_{\Omega} \psi(0, \mathbf{x}) \sigma_0(\mathbf{x}) d\mathbf{x}.$$

Hence,

$$\begin{aligned} - \int_{\Omega} \psi(0, \mathbf{x}) \sigma_0(\mathbf{x}) d\mathbf{x} - \int_{[0, \tau] \times \Omega} \partial_t \mathbf{F}_{(t)}(\mathbf{x}) \cdot (\nabla \psi)(t, \mathbf{F}_{(t)}(\mathbf{x})) \sigma_0(\mathbf{x}) dt d\mathbf{x} \\ = \int_{[0, \tau] \times \Omega} \partial_t \psi(t, \mathbf{X}) \sigma(t, \mathbf{X}) dt d\mathbf{X}. \end{aligned}$$

If we make the change of variables  $\mathbf{X} = \mathbf{F}_{(t)}(\mathbf{x})$  in the second integral above, then, by (ii), (iii) in Definition 5.2 and by (2.5), we have that

$$\begin{aligned} - \int_{\Omega} \psi(0, \mathbf{x}) \sigma_0(\mathbf{x}) d\mathbf{x} - \int_{[0, \tau] \times \Omega} (\partial_t \mathbf{F}_{(t)})(\mathbf{F}_{(t)}^*(\mathbf{X})) \cdot \nabla \psi(t, \mathbf{X}) \sigma(t, \mathbf{X}) dt d\mathbf{X} \\ = \int_{[0, \tau] \times \Omega} \partial_t \psi(t, \mathbf{X}) \sigma(t, \mathbf{X}) dt d\mathbf{X}. \end{aligned}$$

Then, rearranging and using the definition of  $\mathbf{u}$  in (5.16), we obtain

$$\int_{[0, \tau] \times \Omega} \{ \partial_t \psi(t, \mathbf{X}) + \mathbf{u}(t, \mathbf{X}) \cdot \nabla \psi(t, \mathbf{X}) \} \sigma(t, \mathbf{X}) dt d\mathbf{X} + \int_{\Omega} \psi(0, \mathbf{x}) \sigma_0(\mathbf{x}) d\mathbf{x} = 0.$$

Changing notations  $\mathbf{X}$  to  $\mathbf{x}$  gives us (5.9).

We now prove that (5.8) also holds. By the properties of  $\mathbf{F}$  and  $\mathbf{T}$  in Definition 5.2, we have that  $\mathbf{Z}(t, \mathbf{x})$  as defined in (5.12) satisfies  $\mathbf{Z} \in L^\infty([0, \tau] \times \overline{\Omega})$ . Also, applying the

definition of  $\mathbf{Z}(t, \mathbf{x})$  in (5.12) to equation (5.14) gives

$$\begin{aligned} \int_{[0, \tau) \times \Omega} \{ \mathbf{T}(t, \mathbf{F}_{(t)}(\mathbf{x})) \cdot \partial_t \varphi(t, \mathbf{x}) + f_{\text{cor}} \mathbf{e}_3 \times [\mathbf{T}(t, \mathbf{F}_{(t)}(\mathbf{x})) - \mathbf{F}_{(t)}(\mathbf{x})] \cdot \varphi(t, \mathbf{x}) \} \sigma_0(\mathbf{x}) dt d\mathbf{x} \\ + \int_{\Omega} [\mathbf{T}_0(\mathbf{x}) \cdot \varphi(0, \mathbf{x})] \sigma_0(\mathbf{x}) d\mathbf{x} = 0, \end{aligned} \quad (5.17)$$

for any  $\varphi \in C_c^1([0, \tau) \times \Omega)$ . Now, since  $\Omega$  is a bounded set and  $\mathbf{F}_{(t)} \# \sigma_0(\cdot) = \sigma(t, \cdot)$  for all  $t \in [0, \tau)$ , equation (2.5) allows us to make the change of variables  $\mathbf{X} = \mathbf{F}_{(t)}(\mathbf{x})$  in the first integral of (5.17). Thus, by (iii) of Definition 5.2, we have that  $\mathbf{x} = \mathbf{F}_{(t)}^*(\mathbf{X})$  for  $\sigma - a.e.$   $\mathbf{x} \in \Omega$  for every  $t \in [0, \tau)$  and then, from (5.17), we obtain

$$\begin{aligned} \int_{[0, \tau) \times \Omega} \{ \mathbf{T}(t, \mathbf{X}) \cdot \partial_t \varphi(t, \mathbf{F}_{(t)}^*(\mathbf{X})) + f_{\text{cor}} \mathbf{e}_3 \times [\mathbf{T}(t, \mathbf{X}) - \mathbf{X}] \cdot \varphi(t, \mathbf{F}_{(t)}^*(\mathbf{X})) \} \sigma(t, \mathbf{X}) dt d\mathbf{X} \\ + \int_{\Omega} [\mathbf{T}_0(\mathbf{x}) \cdot \varphi(0, \mathbf{x})] \sigma_0(\mathbf{x}) d\mathbf{x} = 0, \end{aligned} \quad (5.18)$$

for any  $\varphi \in C_c^1([0, \tau) \times \Omega)$ . We now show that (5.18) also holds for all  $\varphi$  such that

$$\begin{aligned} \varphi &\in L^\infty([0, \tau) \times \Omega), \\ \partial_t \varphi &\in L^\infty([0, \tau) \times \Omega), \\ \text{supp}(\varphi) &\subset [0, \tau - \epsilon] \times \overline{\Omega} \quad \text{for some } \epsilon > 0 \end{aligned} \quad (5.19)$$

In order to do this, we construct an approximating sequence for such  $\varphi$ . Let us extend  $\varphi$  to  $(-\infty, \infty) \times \Omega$  by defining, for  $\mathbf{x} \in \Omega$ ,  $\varphi(t, \mathbf{x}) = \varphi(-t, \mathbf{x})$  for  $t < 0$  and  $\varphi(t, \cdot) \equiv 0$  for  $t \geq \tau$ . Then, we let  $h > 0$  and define  $\Omega_h = \{\mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \partial\Omega) > h\}$ , where  $\partial\Omega$  denotes the boundary of  $\Omega$ . Thus,  $\varphi \chi_{\Omega_h}$  is now defined on  $\mathbb{R}^1 \times \mathbb{R}^3$ , where  $\chi_{\Omega_h}$  denotes the characteristic function of the set  $\Omega_h$ . Next, let  $j_h(t, \mathbf{x}) = \frac{1}{h^4} j(\frac{|(t, \mathbf{x})|}{h})$ , where  $j(\cdot)$  is a standard mollifier, and let  $k > \frac{1}{\epsilon}$  be an integer. We then have that functions  $\varphi_k = (\varphi \chi_{\Omega_{4h}}) * j_h$ , with  $h = \frac{1}{k} < \epsilon$ , satisfy

$$\varphi_k \in C_c^1([0, \tau) \times \Omega) \quad \text{with} \quad \|\varphi_k, \partial_t \varphi_k\|_{L^\infty([0, \tau) \times \Omega)} \leq C,$$

where  $C$  does not depend on  $k$ , and

$$(\varphi_k, \partial_t \varphi_k) \rightarrow (\varphi, \partial_t \varphi) \quad a.e. \text{ on } [0, \tau) \times \Omega \text{ as } k \rightarrow \infty.$$

Thus, by the dominated convergence theorem,

$$\lim_{k \rightarrow \infty} \int_{[0, \tau) \times \Omega} (\varphi_k, \partial_t \varphi_k)(t, \mathbf{x}) \sigma_0(\mathbf{x}) dt d\mathbf{x} = \int_{[0, \tau) \times \Omega} (\varphi, \partial_t \varphi)(t, \mathbf{x}) \sigma_0(\mathbf{x}) dt d\mathbf{x}.$$

Also, since  $\varphi \in L^\infty([0, \tau) \times \Omega)$  by (5.19) and  $\mathbf{F}_{(t)}^* \# \sigma(t, \cdot) = \sigma_0(\cdot)$ , we have from (2.5) that

$$\int_{[0, \tau) \times \Omega} (\varphi_k, \partial_t \varphi_k)(t, \mathbf{x}) \sigma_0(\mathbf{x}) dt d\mathbf{x} = \int_{[0, \tau) \times \Omega} (\varphi_k, \partial_t \varphi_k)(t, \mathbf{F}_{(t)}^*(\mathbf{X})) \sigma(t, \mathbf{X}) dt d\mathbf{X}$$

and

$$\int_{[0, \tau) \times \Omega} (\varphi, \partial_t \varphi)(t, \mathbf{x}) \sigma_0(\mathbf{x}) dt d\mathbf{x} = \int_{[0, \tau) \times \Omega} (\varphi, \partial_t \varphi)(t, \mathbf{F}_{(t)}^*(\mathbf{X})) \sigma(t, \mathbf{X}) dt d\mathbf{X}.$$

Hence,

$$\lim_{k \rightarrow \infty} \int_{[0, \tau) \times \Omega} (\varphi_k, \partial_t \varphi_k)(t, \mathbf{F}_{(t)}^*(\mathbf{X})) \sigma(t, \mathbf{X}) dt d\mathbf{X} = \int_{[0, \tau) \times \Omega} (\varphi, \partial_t \varphi)(t, \mathbf{F}_{(t)}^*(\mathbf{X})) \sigma(t, \mathbf{X}) dt d\mathbf{X}.$$

Then, since  $\Omega$  is bounded,  $\mathbf{T} \in L^\infty([0, \tau) \times \Omega)$  and (5.18) holds for each  $\varphi_k$ , it follows that (5.18) holds for  $\varphi$  satisfying (5.19). Now, let

$$\varphi(t, \mathbf{x}) = \eta(t, \mathbf{F}_{(t)}(\mathbf{x})), \quad (5.20)$$

where  $\eta \in C_c^1([0, \tau) \times \Omega)$ . Then, since  $\text{supp}(\eta) \subset [0, \tau - \epsilon] \times \overline{\Omega}$  and  $\partial_t \mathbf{F} \in L^\infty([0, \tau) \times \Omega)$  by our additional regularity property, we have that  $\varphi$  satisfies the conditions in (5.19). Therefore, (5.18) holds for  $\varphi$  as defined in (5.20). We also see from (5.20) and use of the chain rule that

$$\partial_t \varphi(t, \mathbf{x}) = (\partial_t \eta)(t, \mathbf{F}_{(t)}(\mathbf{x})) + \partial_t \mathbf{F}_{(t)}(\mathbf{x}) \cdot (\nabla \eta)(t, \mathbf{F}_{(t)}(\mathbf{x})).$$

Thus, by property (iii) of Definition 5.2, we have

$$\varphi(t, \mathbf{F}_{(t)}^*(\mathbf{X})) = \eta(t, \mathbf{X})$$

and

$$\partial_t \varphi(t, \mathbf{F}_{(t)}^*(\mathbf{X})) = \partial_t \eta(t, \mathbf{X}) + [(\partial_t \mathbf{F})(t, \mathbf{F}_{(t)}^*(\mathbf{X}))] \cdot \nabla \eta(t, \mathbf{X})$$

for  $\sigma - a.e.$   $\mathbf{X} \in \Omega$  for every  $t \in [0, \tau)$ . Then, inserting  $\varphi(t, \mathbf{x}) = \eta(t, \mathbf{F}_{(t)}(\mathbf{x}))$  into (5.18) gives

$$\begin{aligned} & \int_{[0, \tau) \times \Omega} \{ \mathbf{T}(t, \mathbf{X}) \cdot [\partial_t \eta(t, \mathbf{X}) + [(\partial_t \mathbf{F})(t, \mathbf{F}_{(t)}^*(\mathbf{X}))] \cdot \nabla \eta(t, \mathbf{X})] + f_{\text{cor}} \mathbf{e}_3 \times [\mathbf{T}(t, \mathbf{X}) - \mathbf{X}] \cdot \eta(t, \mathbf{X}) \} \sigma(t, \mathbf{X}) dt d\mathbf{X} \\ & + \int_{\Omega} \mathbf{T}_0(\mathbf{x}) \cdot \eta(0, \mathbf{F}_{(0)}(\mathbf{x})) \sigma_0(\mathbf{x}) d\mathbf{x} = 0, \end{aligned}$$

and, using the definition of  $\mathbf{u}$  in (5.16) together with property (i) of Definition 5.2, we obtain

$$\begin{aligned} & \int_{[0, \tau) \times \Omega} \{ \mathbf{T}(t, \mathbf{X}) \cdot [\partial_t \eta(t, \mathbf{X}) + (\mathbf{u}(t, \mathbf{X}) \cdot \nabla) \eta(t, \mathbf{X})] + f_{\text{cor}} \mathbf{e}_3 \times [\mathbf{T}(t, \mathbf{X}) - \mathbf{X}] \cdot \eta(t, \mathbf{X}) \} \sigma(t, \mathbf{X}) dt d\mathbf{X} \\ & + \int_{\Omega} \mathbf{T}_0(\mathbf{x}) \cdot \eta(0, \mathbf{x}) \sigma_0(\mathbf{x}) d\mathbf{x} = 0. \end{aligned}$$

Finally, changing notations  $\mathbf{X}$  to  $\mathbf{x}$  and  $\eta$  to  $\varphi$  gives (5.8), proving statement (i). Then, statement (ii) follows directly from (i).  $\square$

We now state the main result, which we will prove in Sections 6 and 7:

**Theorem 5.5.** *Let  $\Omega \subset \mathbb{R}^3$  be an open bounded convex set. Assume that*

$$\mathbf{T}_0 \# \sigma_0 \in L^r(\Lambda_0)$$

*for  $r \in (1, \infty)$ , where  $\Lambda_0 \subset \mathbb{R}^3$  is compact. Then, for any  $\tau > 0$ , there exists a weak Lagrangian solution  $(\mathbf{F}, \mathbf{T}, \sigma)$  of (5.1)-(5.5) in  $[0, \tau) \times \Omega$ , where  $\mathbf{T} \in L^\infty([0, \tau) \times \Omega)$  and (5.10) is satisfied for any  $q \in [1, \infty)$ . Moreover, the function  $\mathbf{Z}(t, \mathbf{x})$  defined by (5.12) satisfies  $\mathbf{Z}(\cdot, \mathbf{x}) \in W^{1, \infty}([0, \tau))$  for  $\sigma_0 - a.e.$   $\mathbf{x} \in \Omega$ , and (5.13) is satisfied, in addition to the weak form (5.14), in the following sense:*

$$\begin{aligned} \partial_t \mathbf{Z}(t, \mathbf{x}) &= f_{\text{cor}} \mathbf{e}_3 \times [\mathbf{Z}(t, \mathbf{x}) - \mathbf{F}(t, \mathbf{x})] \quad \text{for } \sigma_0 - a.e. \mathbf{x} \in \Omega \text{ and every } t \in [0, \tau), \\ \mathbf{Z}(0, \mathbf{x}) &= \mathbf{T}_0(\mathbf{x}) \quad \text{for } \sigma_0 - a.e. \mathbf{x} \in \Omega. \end{aligned} \quad (5.21)$$

To prove this theorem, we show the existence of a Lagrangian flow map  $\Phi$  in dual variables. We then define  $\mathbf{F}$  in terms of  $\Phi$ .

Throughout the next two sections, we will use  $\sigma$  and  $\nu$  to denote the measures  $\sigma(t, \cdot)$  and  $\nu(t, \cdot)$  considered at some fixed time  $t \in [0, \tau)$ . Thus, if we write  $\nu - a.e. \mathbf{y} \in \mathbb{R}^3$ , for example, then we mean  $\nu(t, \cdot) - a.e. \mathbf{y} \in \mathbb{R}^3$  for a fixed  $t \in [0, \tau)$ .

## 6 Existence of the Lagrangian flow map in dual space

In this section we use the results of [1] to show existence of the Lagrangian flow map in dual space.

Let  $\Omega, \Lambda$  be as in Theorem 4.1, let  $\mathbf{T}_0, \sigma_0$  be as in (5.5) and let  $\nu_0 = \mathbf{T}_0 \# \sigma_0$ . Then, by Theorem 4.1 there exists a solution  $(\mathbf{T}, \mathbf{S}, \sigma)$  of the system (3.14)-(3.19), with initial data  $\nu_0$ , satisfying all assertions of Theorem 4.1. We use Theorem 3.4, Theorem 3.5, and the definition of  $\mathbf{w}$  in (3.15) to see that the dual space velocity can be written as

$$\mathbf{w} = f_{\text{cor}}^{-1} y_3 \mathbf{e}_3 \times \nabla g_0(\mathbf{y}) \quad (6.1)$$

where  $g_0$  is a solution of the Kantorovich dual problem (3.23). Thus, the vector field  $\mathbf{w}$  is divergence free. Furthermore, since the potential temperature  $y_3$  is assumed to be bounded, we can use (6.1) to obtain the following properties of  $\mathbf{w}$ :

- by the semi-concavity of  $g_0$  in  $\Lambda$ , it follows that  $\mathbf{w}(t, \cdot) \in BV_{\text{loc}}(\mathbb{R}^3)$  for  $a.e. t \in (0, \tau)$ ;
- by Theorem 4.1 we have that  $\mathbf{w} \in L^\infty([0, \tau) \times \Lambda)$ .

Since  $\nu$  has compact support in  $[0, \tau] \times \mathbb{R}^3$ , we can modify  $\mathbf{w}$  away from  $\Lambda$  with a standard smooth cut off function  $\zeta \in C^\infty(\mathbb{R}^3)$  so that the modified function  $\tilde{\mathbf{w}} := \zeta \mathbf{w}$  satisfies

$$\tilde{\mathbf{w}} \in L^\infty([0, \tau) \times \mathbb{R}^3), \quad \tilde{\mathbf{w}}(t, \cdot) \in BV_{\text{loc}}(\mathbb{R}^3) \text{ for } a.e. t \in (0, \tau), \quad (6.2)$$

$$\nabla \cdot \tilde{\mathbf{w}}(t, \cdot) = 0 \text{ in } \mathbb{R}^3 \text{ for every } t \in [0, \tau). \quad (6.3)$$

and

$$\mathbf{w} = \tilde{\mathbf{w}} \quad \text{in } \Lambda. \quad (6.4)$$

These conditions enable us to apply the theory of [1] to the transport equation (3.14) with  $\mathbf{w}$  replaced by our modified velocity  $\tilde{\mathbf{w}}$ :

**Lemma 6.1.** *There exists a unique locally bounded Borel measurable map  $\Phi : [0, \tau) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$  satisfying*

- (i)  $\Phi(\cdot, \mathbf{y}) \in W^{1, \infty}([0, \tau))$  for  $\nu_0 - a.e. \mathbf{y} \in \mathbb{R}^3$ ;
- (ii)  $\Phi(0, \mathbf{y}) = \mathbf{y}$  for  $\nu_0 - a.e. \mathbf{y} \in \mathbb{R}^3$ ;
- (iii) for  $\nu_0 - a.e. \mathbf{y} \in \mathbb{R}^3$ ,
$$\partial_t \Phi(t, \mathbf{y}) = \tilde{\mathbf{w}}(t, \Phi(t, \mathbf{y})); \quad (6.5)$$
- (iv) there exists a Borel map  $\Phi^* : [0, \tau) \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$  such that, for every  $t \in (0, \tau)$ , the map  $\Phi_{(t)}^* : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is Lebesgue-measure preserving, and such that  $\Phi_{(t)}^* \circ \Phi_{(t)}(\mathbf{y}) = \Phi_{(t)} \circ \Phi_{(t)}^*(\mathbf{y}) = \mathbf{y}$  for  $\nu - a.e. \mathbf{y} \in \mathbb{R}^3$ ;
- (v)  $\Phi(t, \cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  is a Lebesgue-measure preserving map for every  $t \in [0, \tau)$ .

*Proof.* The proof is essentially identical to that of [6, Lemma 2.8].  $\square$

We now show that the image of the flow map  $\Phi$  is contained in  $\Lambda$ , and therefore corresponds to the velocity field  $\mathbf{w}$ .

**Lemma 6.2.** *Let  $\Lambda$  be as in Theorem 4.1. Let  $\Phi$  be the map defined in Lemma 6.1 and let  $\mathbf{w}$  be defined as in (3.15). Then*

$$\Phi(t, \mathbf{y}) \subset \Lambda \quad \text{for } \nu_0 - \text{a.e. } \mathbf{y} \in \mathbf{T}_0(\Omega) \text{ and every } t \in [0, \tau]. \quad (6.6)$$

In particular,

$$\partial_t \Phi(t, \mathbf{y}) = \mathbf{w}(t, \Phi(t, \mathbf{y})) \quad \text{for } \nu_0 - \text{a.e. } \mathbf{y} \in \mathbf{T}_0(\Omega) \text{ and every } t \in [0, \tau]. \quad (6.7)$$

*Proof.* Firstly note that, since  $\mathbf{T}_0 \# \sigma_0 = \nu_0$  and the measures  $\sigma_0$  and  $\nu_0$  are contained within  $\Omega$  and  $\Lambda$  respectively, we may assume

$$\mathbf{T}_0 : \Omega \rightarrow \Lambda. \quad (6.8)$$

Therefore, for  $\sigma_0 - \text{a.e. } \mathbf{x} \in \Omega$ , we have that  $\mathbf{T}_0(\mathbf{x}) \in \Lambda$ .

We begin with the vertical component of  $\Phi$ ;  $\Phi_3$ . From (6.4) and (6.5), we have that  $\partial_t \Phi_3(t, \mathbf{y}) = \tilde{w}_3(t, \Phi(t, \mathbf{y})) = 0$  for all  $\mathbf{y} \in \mathbb{R}^3$  and every  $t \in [0, \tau]$ . Therefore, we have  $\Phi_3(t, \mathbf{y}) = \Phi_3(0, \mathbf{y}) = y_3$  for  $\nu_0 - \text{a.e. } \mathbf{y} \in \mathbb{R}^3$  and every  $t \in [0, \tau]$ , where we have used Lemma 6.1 (ii). We therefore conclude that  $\delta < \Phi_3(t, \mathbf{y}) < \frac{1}{\delta}$  for  $\nu_0 - \text{a.e. } \mathbf{y} \in \mathbf{T}_0(\Omega)$  and every  $t \in [0, \tau]$ , where  $\delta$  is as in Remark 3.1.

The proof that the horizontal components of  $\Phi$  stay inside  $\Lambda$  proceeds in a similar way to that of [6, Lemma 2.9].  $\square$

Now, we wish to prove that when  $(\nu, \mathbf{T})$  is a weak solution of (3.14)-(3.19) then  $\nu$  is a weak Lagrangian solution of the transport equation (3.14), i.e.  $\nu$  satisfies the property  $\nu = \Phi \# \nu_0$ .

First, we state the following result from [3]:

**Proposition 6.3.** *Let  $\nu(t, \cdot)$  be narrowly continuous Borel probability measures solving the continuity equation*

$$\partial_t \nu + \nabla \cdot (\mathbf{v} \nu) = 0$$

*with respect to a vector field  $\mathbf{v}$  satisfying*

$$\int_{[0, \tau) \times \mathbb{R}^3} |\mathbf{v}| \nu \, dt d\mathbf{y} < +\infty, \quad (6.9)$$

$$\int_0^\tau \left[ \sup_B |\mathbf{v}| + \text{Lip}(\mathbf{v}, B) \right] dt < +\infty, \quad (6.10)$$

*for every compact set  $B \subset \mathbb{R}^3$ . Then, for  $\nu_0 - \text{a.e. } \mathbf{y} \in \mathbb{R}^3$  the characteristic system*

$$\Phi(0, \mathbf{y}) = \mathbf{y}, \quad \frac{d}{dt} \Phi(t, \mathbf{y}) = \mathbf{v}(t, \Phi(t, \mathbf{y})) \quad (6.11)$$

*admits a globally defined solution  $\Phi(t, \mathbf{y})$  in  $[0, \tau)$  and*

$$\nu(t, \cdot) = \Phi(t, \cdot) \# \nu_0(\cdot) \quad \text{for all } t \in [0, \tau). \quad (6.12)$$

*Proof.* See [3, Proposition 8.1.8].  $\square$

Note that  $\tilde{\mathbf{w}}$  is not Lipschitz continuous and so does not satisfy the regularity assumption (6.10). As such, we approximate  $\tilde{\mathbf{w}}$  in order to prove the following

**Proposition 6.4.** *Let  $\Omega$ ,  $\Lambda$ ,  $r$ ,  $\mathbf{T}_0$  be as in Theorem 4.1, and let  $(\nu, \mathbf{T})$  be the weak solution of (3.14)-(3.19) as constructed in Theorem 4.1. Let  $\Phi$  be the regular Lagrangian flow of  $\tilde{\mathbf{w}}$  defined in Lemma 6.1. Then, for every  $t \in [0, \tau]$ ,*

$$\nu(t, \cdot) = \Phi(t, \cdot) \# \nu_0(\cdot). \quad (6.13)$$

Moreover, for every  $t \in [0, \tau]$ ,

$$\nu(t, \mathbf{y}) = \nu_0(\Phi^*(t, \mathbf{y})) \quad \text{for } \nu - a.e. \quad \mathbf{y} \in \mathbb{R}^3, \quad (6.14)$$

where the map  $\Phi^*$  is defined in Lemma 6.1 (iv).

*Proof.* Notice first that, by (6.2) and the compact support of  $\tilde{\mathbf{w}}$ , we have

$$\int_{[0, \tau] \times \mathbb{R}^3} |\tilde{\mathbf{w}}|^p \nu \, dt d\mathbf{y} < +\infty, \quad (6.15)$$

where  $p \geq 1$ .

Let  $(j_\epsilon) \subset C^\infty(\mathbb{R}^3)$  be a family of strictly positive mollifiers in the  $\mathbf{y}$  variable and set  $\nu_\epsilon := \nu * j_\epsilon$ ,  $\mathbf{k}_\epsilon = (\tilde{\mathbf{w}}\nu) * j_\epsilon$ ,  $\tilde{\mathbf{w}}_\epsilon := \frac{\mathbf{k}_\epsilon}{\nu_\epsilon}$ . Then, since the transport equation (3.14) is satisfied with  $\mathbf{w}$  replaced by our modified velocity  $\tilde{\mathbf{w}}$ , and since (6.15) holds, we may use [3, Lemma 8.1.9] to conclude that  $\nu_\epsilon$  is a continuous solution of

$$\partial_t \nu_\epsilon + \nabla \cdot (\tilde{\mathbf{w}}_\epsilon \nu_\epsilon) = 0$$

and that  $\tilde{\mathbf{w}}_\epsilon$  satisfy the regularity assumptions (6.10) and the uniform integrability bounds

$$\int_{\mathbb{R}^3} |\tilde{\mathbf{w}}_\epsilon|^p \nu_\epsilon \, d\mathbf{y} \leq \int_{\mathbb{R}^3} |\tilde{\mathbf{w}}|^p \nu \, d\mathbf{y},$$

for all  $t \in (0, \tau)$ . Moreover,

$$\nu_\epsilon \rightarrow \nu, \quad \mathbf{k}_\epsilon \rightarrow \tilde{\mathbf{w}}\nu \quad (6.16)$$

narrowly and

$$\lim_{\epsilon \rightarrow 0^+} \|\tilde{\mathbf{w}}_\epsilon\|_{L^p(\nu_\epsilon; \mathbb{R}^3)} = \|\tilde{\mathbf{w}}\|_{L^p(\nu; \mathbb{R}^3)},$$

for all  $t \in (0, \tau)$ .

Thus, from Proposition 6.3, we obtain

$$\nu_\epsilon(t, \cdot) = \Phi_\epsilon(t, \cdot) \# \nu_\epsilon(0, \cdot)$$

for all  $t \in [0, \tau]$ , where  $\Phi_\epsilon$  is a globally defined solution of

$$\Phi(0, \mathbf{y}) = \mathbf{y}, \quad \frac{d}{dt} \Phi(t, \mathbf{y}) = \tilde{\mathbf{w}}_\epsilon(t, \Phi(t, \mathbf{y})).$$

Thus, by Theorem 4.1 (i) and the definition of  $\nu_\epsilon$ , we may apply (2.5) to obtain for any,  $t > 0$  and any  $\varphi \in C_c(\mathbb{R}^3)$ ,

$$\int_{\mathbb{R}^3} \varphi(\Phi_\epsilon(t, \mathbf{y})) \nu_\epsilon(0, \mathbf{y}) \, d\mathbf{y} = \int_{\mathbb{R}^3} \varphi(\mathbf{Y}) \nu_\epsilon(t, \mathbf{Y}) \, d\mathbf{Y}.$$



Passing to the limit  $\epsilon \rightarrow 0$  and combining (6.16) with [1, Theorem 6.6] and the dominated convergence theorem, we conclude that

$$\int_{\mathbb{R}^3} \varphi(\Phi(t, \mathbf{y})) \nu_0(\mathbf{y}) d\mathbf{y} = \int_{\mathbb{R}^3} \varphi(\mathbf{Y}) \nu(t, \mathbf{Y}) d\mathbf{Y}, \quad (6.17)$$

where  $\Phi$  denotes the regular Lagrangian flow relative to  $\tilde{\mathbf{w}}$  (and therefore, by Lemma 6.2, relative to  $\mathbf{w}$  in  $\Lambda$ ) given by Lemma 6.1. By (2.5), this implies (6.13). Since  $\Phi$  is a measure preserving map, we use Lemma 6.1 (iv) to conclude that the left-hand side of (6.17) is equal to

$$\int_{\mathbb{R}^3} \varphi(\mathbf{Y}) \nu_0(\Phi^*(t, \mathbf{Y})) d\mathbf{Y},$$

and now (6.17) implies (6.14). □

## 7 Lagrangian flow in physical space

Throughout this section we will assume that  $\Omega, \Lambda, r, \mathbf{T}_0, \mathbf{T}, \nu, \mathbf{w}, \tilde{\mathbf{w}}, \Phi$  are as in Proposition 6.4. Note that, by Theorem 4.1, we can apply (2.5) to  $\sigma_0, \sigma, \nu_0, \nu$  throughout this section.

We now perform the last step of the analysis and prove the existence of a Lagrangian flow  $\mathbf{F} : [0, \tau) \times \Omega \rightarrow \Omega$  in the physical space. Indeed, we define  $\mathbf{F}_{(t)} : \Omega \rightarrow \Omega$  for  $t \in [0, \tau)$  as

$$\mathbf{F}_{(t)} := \mathbf{S}_{(t)} \circ \Phi_{(t)} \circ \mathbf{T}_0, \quad (7.1)$$

where  $\mathbf{T}_0$  is as in (5.5),  $\mathbf{S}_{(t)}$  is the inverse of  $\mathbf{T}_{(t)}$  (see Theorem 3.5) and  $\Phi_{(t)}$  is the Lagrangian flow in dual space constructed in Lemma 6.1. To justify this definition, we prove the following lemma:

**Lemma 7.1.** *For any  $t \in [0, \tau)$ , the right hand side of (7.1) is defined  $\sigma_0 - a.e.$  in  $\Omega$ . The map  $\mathbf{F} : [0, \tau) \times \Omega \rightarrow \Omega$  defined by (7.1) is Borel.*

*Proof.* Since  $\mathbf{T}_0$  exists and is unique  $\sigma_0 - a.e.$  in  $\Omega$ , we have that  $\mathbf{T}_0$  exists and is unique on  $\Omega \setminus N_0^1$  where  $N_0^1$  is a Borel subset of  $\Omega$  with  $\sigma_0[N_0^1] = 0$ . Also, since  $\mathbf{S}$  exists and is unique  $\nu - a.e.$  in  $\Lambda$  for every  $t \in [0, \tau)$ , we have that  $\mathbf{S}$  exists and is unique on  $\Lambda \setminus N^2$  for every  $t \in [0, \tau)$ , where  $N^2$  is a Borel subset of  $\Lambda$  with  $\nu[N^2] = 0$ . Then, the right-hand side of (7.1) is defined for all

$$\mathbf{x} \in \Omega \setminus (N_0^1 \cup M),$$

where

$$M = \{\mathbf{X} \in (\Omega \setminus N_0^1) : \Phi_{(t)}(\mathbf{T}_0(\mathbf{X})) \in N^2\}.$$

Note that, from its definition,  $M$  is a Borel set.

It remains to prove that  $\sigma_0[M] = 0$  for every  $t \in [0, \tau)$

Fix  $t \in [0, \tau)$ . Then, using that  $\mathbf{T}_0 \# \sigma_0 = \nu_0$  and thus  $\mathbf{T}_0 \# \sigma_0 = \nu_0$  for all  $\mathbf{x} \in \Omega \setminus N_0^1$ , and using (6.13) as well as Lemma 6.1 (iv), we can apply (2.5) and compute

$$\sigma_0[M_{(t)}] = \sigma_0 \left[ \left\{ \mathbf{x} \in \Omega \setminus N_0^1 : \mathbf{T}_0(\mathbf{x}) \in \Phi_{(t)}^{-1}(N^2) \right\} \right] \quad (7.2)$$

$$= \int_{\mathbf{T}_0^{-1}(\Phi_{(t)}^{-1}(N^2))} \sigma_0(\mathbf{x}) d\mathbf{x} = \int_{\Phi_{(t)}^{-1}(N^2)} \nu_0(\mathbf{y}) d\mathbf{y} \quad (7.3)$$

$$= \int_{N^2} \nu(\mathbf{y}) d\mathbf{y} = 0.$$

Thus, we can define  $\mathbf{F} : [0, \tau) \times \Omega \rightarrow \Omega$  by (7.1). Then, by Lemma 6.1,  $\mathbf{F}$  is a Borel mapping.  $\square$

It remains to prove that, if  $\mathbf{F}$  is defined by (7.1), then  $(\mathbf{F}, \mathbf{T}, \sigma)$  is a weak Lagrangian solution of (5.1)-(5.5) in the sense of Definition 5.2. We begin by showing that the initial condition for the flow in Definition 5.2 (i) is satisfied.

**Proposition 7.2.** *Let  $\mathbf{F}$  be defined as in (7.1). Then,  $\mathbf{F}(0, \mathbf{x}) = \mathbf{x}$  for  $\sigma_0$  - a.e.  $\mathbf{x} \in \Omega$ .*

*Proof.* By (7.1) we have that  $\mathbf{F}_{(0)}(\mathbf{x}) = \mathbf{S}_{(0)} \circ \Phi_{(0)} \circ \mathbf{T}_0(\mathbf{x})$  for all  $\mathbf{x} \in \Omega \setminus N_0$  where  $N_0$  is a Borel set with  $\sigma_0[N_0] = 0$ .

By Lemma 3.5, there exist Borel sets  $N_1 \subset \Omega$ ,  $N_2 \subset \Lambda$  with  $\sigma_0[N_1] = \nu_0[N_2] = 0$  such that  $\mathbf{T}_0$  exists and is unique in  $\Omega \setminus N_1$  and  $\mathbf{S}_{(0)}$  exists and is unique in  $\Lambda \setminus N_2$ . Moreover, if  $\mathbf{x} \in \Omega \setminus [N_1 \cup \mathbf{T}_0^{-1}(N_2)]$ , then  $\mathbf{S}_{(0)} \circ \mathbf{T}_0(\mathbf{x}) = \mathbf{x}$ . Also, by Lemma 6.1 (ii), we have that  $\Phi_{(0)}(\mathbf{y}) = \mathbf{y}$  in  $\Lambda \setminus N_3$ , where  $N_3$  is a Borel set with  $\nu_0[N_3] = 0$ .

Therefore,  $\mathbf{F}_{(0)}(\mathbf{x}) = \mathbf{x}$  for all  $\mathbf{x} \in \Omega \setminus [N_0 \cup N_1 \cup \mathbf{T}_0^{-1}(N_2 \cup N_3)]$ . We must now show that  $\sigma_0[\Omega \cap \mathbf{T}_0^{-1}(N_2 \cup N_3)] = 0$ .

We have that  $\nu_0[N_2 \cup N_3] = 0$ . Then, since  $\mathbf{T}_0 \# \sigma_0 = \nu_0$ , we obtain

$$\sigma_0[\Omega \cap \mathbf{T}_0^{-1}(N_2 \cup N_3)] = \int_{\mathbf{T}_0^{-1}(N_2 \cup N_3)} \sigma_0(\mathbf{x}) d\mathbf{x} = \int_{N_2 \cup N_3} \nu_0(\mathbf{y}) d\mathbf{y} = \nu_0[N_2 \cup N_3] = 0.$$

$\square$

Next, we prove that property (ii) of Definition 5.2 is satisfied.

**Proposition 7.3.** *For every  $t > 0$ , the map  $\mathbf{F}_{(t)} : \Omega \rightarrow \Omega$  as defined in (7.1) satisfies  $\mathbf{F}_{(t)} \# \sigma_0 = \sigma$ .*

*Proof.* In order to prove that  $\mathbf{F}_{(t)} \# \sigma_0 = \sigma$ , we must show that, for any  $\varphi \in C(\mathbb{R}^3)$ ,

$$\int_{\Omega} \varphi(\mathbf{F}_{(t)}(\mathbf{x})) \sigma_0(\mathbf{x}) d\mathbf{x} = \int_{\Omega} \varphi(\mathbf{X}) \sigma(\mathbf{X}) d\mathbf{X}.$$

Then the result will follow from (2.5).

Let  $\varphi \in C(\mathbb{R}^3)$ . From the definition of  $\mathbf{F}_{(t)}$  in (7.1) we have

$$\int_{\Omega} \varphi(\mathbf{F}_{(t)}(\mathbf{x})) \sigma_0(\mathbf{x}) d\mathbf{x} = \int_{\Omega} \varphi \circ \mathbf{S}_{(t)} \circ \Phi_{(t)} \circ \mathbf{T}_0(\mathbf{x}) \sigma_0(\mathbf{x}) d\mathbf{x}.$$

Then, using  $\mathbf{T}_0 \# \sigma_0 = \nu_0$  we apply (2.5) to obtain

$$\int_{\Omega} \varphi \circ \mathbf{S}_{(t)} \circ \Phi_{(t)} \circ \mathbf{T}_0(\mathbf{x}) \sigma_0(\mathbf{x}) d\mathbf{x} = \int_{\Lambda} \varphi \circ \mathbf{S}_{(t)} \circ \Phi_{(t)}(\mathbf{y}) \nu_0(\mathbf{y}) d\mathbf{y},$$

since  $\varphi \circ \mathbf{S}_{(t)} \circ \Phi_{(t)} \in L^\infty(\Lambda)$ . Then, since  $\varphi \circ \mathbf{S}_{(t)} \in L^\infty(\Lambda)$  we can use (6.13) and apply (2.5) to get

$$\int_{\Lambda} \varphi \circ \mathbf{S}_{(t)} \circ \Phi_{(t)}(\mathbf{y}) \nu_0(\mathbf{y}) d\mathbf{y} = \int_{\Lambda} \varphi \circ \mathbf{S}_{(t)}(\mathbf{Y}) \nu(\mathbf{Y}) d\mathbf{Y}.$$

Finally, since  $\mathbf{S}_{(t)}$  satisfies  $\mathbf{S}_{(t)} \# \nu = \sigma$ , we have that

$$\int_{\Lambda} \varphi \circ \mathbf{S}_{(t)}(\mathbf{Y}) \nu(\mathbf{Y}) d\mathbf{Y} = \int_{\Omega} \varphi(\mathbf{X}) \sigma(\mathbf{X}) d\mathbf{X}.$$

Thus, we have shown that

$$\int_{\Omega} \varphi(\mathbf{F}_{(t)}(\mathbf{x})) \sigma_0(\mathbf{x}) d\mathbf{x} = \int_{\Omega} \varphi \circ \mathbf{S}_{(t)} \circ \Phi_{(t)} \circ \mathbf{T}_0(\mathbf{x}) \sigma_0(\mathbf{x}) d\mathbf{x} \quad (7.4)$$

$$= \int_{\Lambda} \varphi \circ \mathbf{S}_{(t)} \circ \Phi_{(t)}(\mathbf{y}) \nu_0(\mathbf{y}) d\mathbf{y} \quad (7.5)$$

$$= \int_{\Lambda} \varphi \circ \mathbf{S}_{(t)}(\mathbf{Y}) \nu(\mathbf{Y}) d\mathbf{Y} \quad (7.6)$$

$$= \int_{\Omega} \varphi(\mathbf{X}) \sigma(\mathbf{X}) d\mathbf{X},$$

as required.  $\square$

We now prove that (5.10) holds for all  $q \in [1, \infty)$ .

**Proposition 7.4.** *For any  $t_0 \in [0, \tau)$  and any  $q \in [1, \infty)$ ,*

$$\lim_{t \rightarrow t_0, t \in [0, \tau)} \int_{\Omega} |\mathbf{F}_{(t)}(\mathbf{x}) - \mathbf{F}_{(t_0)}(\mathbf{x})|^q \sigma_0(\mathbf{x}) d\mathbf{x} = 0.$$

*Proof.* By Lemma 7.1 we have that, for any  $t \in [0, \tau)$ , (7.1) holds  $\sigma_0 - a.e.$  in  $\Omega$ . Thus, since  $\mathbf{T}_0 \# \sigma_0 = \nu_0$ , we see that, for any  $t, t_0 \in [0, \tau)$ ,

$$\begin{aligned} \int_{\Omega} |\mathbf{F}_{(t)}(\mathbf{x}) - \mathbf{F}_{(t_0)}(\mathbf{x})|^q \sigma_0(\mathbf{x}) d\mathbf{x} &= \int_{\Omega} |\mathbf{S}_{(t)} \circ \Phi_{(t)} \circ \mathbf{T}_0(\mathbf{x}) - \mathbf{S}_{(t_0)} \circ \Phi_{(t_0)} \circ \mathbf{T}_0(\mathbf{x})|^q \sigma_0(\mathbf{x}) d\mathbf{x} \\ &= \int_{\Lambda} |\mathbf{S}_{(t)} \circ \Phi_{(t)}(\mathbf{y}) - \mathbf{S}_{(t_0)} \circ \Phi_{(t_0)}(\mathbf{y})|^q \nu_0(\mathbf{y}) d\mathbf{y} \end{aligned} \quad (7.8)$$

$$= \int_{\Lambda} |\mathbf{S}_{(t)} \circ \Phi_{(t)}(\mathbf{y}) - \mathbf{S}_{(t_0)} \circ \Phi_{(t)}(\mathbf{y}) + \mathbf{S}_{(t_0)} \circ \Phi_{(t)}(\mathbf{y}) - \mathbf{S}_{(t_0)} \circ \Phi_{(t_0)}(\mathbf{y})|^q \nu_0(\mathbf{y}) d\mathbf{y} \quad (7.9)$$

$$\leq C \int_{\Lambda} |\mathbf{S}_{(t)} \circ \Phi_{(t)}(\mathbf{y}) - \mathbf{S}_{(t_0)} \circ \Phi_{(t)}(\mathbf{y})|^q \nu_0(\mathbf{y}) d\mathbf{y} \quad (7.10)$$

$$+ C \int_{\Lambda} |\mathbf{S}_{(t_0)} \circ \Phi_{(t)}(\mathbf{y}) - \mathbf{S}_{(t_0)} \circ \Phi_{(t_0)}(\mathbf{y})|^q \nu_0(\mathbf{y}) d\mathbf{y} \quad (7.11)$$

$$=: C(I_1 + I_2).$$

Firstly, we show that  $I_1 \rightarrow 0$  as  $t \rightarrow t_0$ . Note that, using (6.14) and Lemma 6.1 (iv) we have that  $\|\nu_{(t)}\|_{L^q(\Lambda)} = \|\nu_0\|_{L^q(\Lambda)}$  for  $t \in (0, \tau)$ , so that  $\|\nu_{(t)}\|_{L^q(\Lambda)}$  is independent of  $t$ . Let  $r$  and  $r'$  be conjugate exponents (i.e.  $\frac{1}{r} + \frac{1}{r'} = 1$ ), with  $1 < r < \infty$ . Then, we can use (6.13) and Hölder's inequality to estimate

$$I_1 = \int_{\Lambda} |\mathbf{S}_{(t)} \circ \Phi_{(t)}(\mathbf{y}) - \mathbf{S}_{(t_0)} \circ \Phi_{(t)}(\mathbf{y})|^q \nu_0(\mathbf{y}) d\mathbf{y} \quad (7.12)$$

$$= \int_{\Lambda} |\mathbf{S}_{(t)}(\mathbf{y}) - \mathbf{S}_{(t_0)}(\mathbf{y})|^q \nu(\mathbf{y}) d\mathbf{y} \quad (7.13)$$

$$\leq \left\{ \int_{\Lambda} |\mathbf{S}_{(t)}(\mathbf{y}) - \mathbf{S}_{(t_0)}(\mathbf{y})|^{qr'} d\mathbf{y} \right\}^{\frac{1}{r'}} \left\{ \int_{\Lambda} |\nu(\mathbf{y})|^r d\mathbf{y} \right\}^{\frac{1}{r}} \quad (7.14)$$

$$= \|\mathbf{S}_{(t)} - \mathbf{S}_{(t_0)}\|_{qr'}^q \|\nu_{(t)}\|_r \quad (7.15)$$

$$= \|\mathbf{S}_{(t)} - \mathbf{S}_{(t_0)}\|_{qr'}^q \|\nu_0\|_r \rightarrow 0 \text{ as } t \rightarrow t_0.$$

Next, we show that  $I_2 \rightarrow 0$  as  $t \rightarrow t_0$ . Since  $\mathbf{S}_{(t)} \in \Omega$  for each  $t$  and for  $\nu - a.e.$   $\mathbf{y}$ , then, by the dominated convergence theorem, it remains to prove that for every  $t_0$ ,

$$\mathbf{S}_{(t_0)} \circ \Phi_{(t)}(\mathbf{y}) - \mathbf{S}_{(t_0)} \circ \Phi_{(t_0)}(\mathbf{y}) \rightarrow 0 \quad \text{as } t \rightarrow t_0 \quad (7.16)$$

for  $\nu - a.e.$   $\mathbf{y} \in \Lambda$ . First we note that, since  $\Phi_{(t)}$  is measure preserving, then it follows from Lemma 6.1 (i), and the fact that  $\tilde{\mathbf{w}} \in L^\infty([0, \tau] \times \mathbb{R}^3)$  by (6.2), that

$$\Phi_{(t)}(\mathbf{y}) \rightarrow \Phi_{(t_0)}(\mathbf{y}) \quad \text{as } t \rightarrow t_0$$

in  $[0, \tau]$  for  $\nu - a.e.$   $\mathbf{y} \in \Lambda$ . If  $\mathbf{y}$  is such a point and if, in addition,  $\Phi_{(t_0)}(\mathbf{y})$  is a point of continuity for  $\mathbf{S}_{(t_0)}$ , then convergence in (7.16) holds at  $\mathbf{y}$ . Since  $\Phi_{(t_0)}$  is measure preserving, it follows that  $\Phi_{(t_0)}(\mathbf{y})$  is a point of continuity for  $\mathbf{S}_{(t_0)}$  for  $\nu - a.e.$   $\mathbf{y}$ . Thus, (7.16) holds for  $\nu - a.e.$   $\mathbf{y} \in \Lambda$ .  $\square$

**Lemma 7.5.** *Let  $\mathbf{Z}$  be defined as in (5.12) with  $\mathbf{F}$  defined as in (7.1). Then, for all  $t \in [0, \tau]$ ,*

$$\mathbf{Z}_{(t)}(\mathbf{x}) = \Phi_{(t)} \circ \mathbf{T}_0(\mathbf{x})$$

for  $\sigma_0 - a.e.$   $\mathbf{x} \in \Omega$ .

*Proof.* Using (5.12), we have that  $\mathbf{Z}_{(t)} = \mathbf{T}_{(t)} \circ \mathbf{F}_{(t)}$ . Therefore, we need to justify the following formal computation:

$$\mathbf{T}_{(t)} \circ \mathbf{F}_{(t)} = \mathbf{T}_{(t)} \circ \mathbf{S}_{(t)} \circ \Phi_{(t)} \circ \mathbf{T}_0 = \Phi_{(t)} \circ \mathbf{T}_0$$

since, by (3.16),  $\mathbf{T}_{(t)} \circ \mathbf{S}_{(t)}$  is the identity on the support of  $\nu$ .

Now we make this argument rigorous. Since  $\mathbf{T}$  exists and is unique  $\sigma - a.e.$  in  $\Omega$  for every  $t \in [0, \tau]$ , we have that  $\mathbf{T}$  exists and is unique in  $\Omega \setminus N^1$  for every  $t \in [0, \tau]$ , where  $N^1$  is a Borel subset of  $\Omega$  with  $\sigma[N^1] = 0$ . Then, by Proposition 7.3 we have that

$$\sigma_0 \left[ \Omega \cap \mathbf{F}_{(t)}^{-1}(N^1) \right] = \int_{\mathbf{F}_{(t)}^{-1}(N^1)} \sigma_0(\mathbf{x}) d\mathbf{x} = \int_{N^1} \sigma(\mathbf{X}) d\mathbf{X} = 0.$$

Now, using Lemma 7.1, we conclude that

$$\mathbf{Z}_{(t)}(\mathbf{x}) = \mathbf{T}_{(t)} \circ \mathbf{S}_{(t)} \circ \Phi_{(t)} \circ \mathbf{T}_0(\mathbf{x})$$

for  $\mathbf{x} \in \Omega \setminus \tilde{N}$  where  $\sigma_0[\tilde{N}] = 0$ .

Let  $\tilde{M} = \{\mathbf{y} \in \Lambda : \mathbf{T}(t, \mathbf{S}(t, \mathbf{y})) \neq \mathbf{y}\}$ , Then  $\tilde{M}$  is a Borel set. Now, the proof of the lemma will be completed if we show that

$$\sigma_0 \left[ \{\mathbf{x} \in \Omega \setminus \tilde{N} : \Phi_{(t)} \circ \mathbf{T}_0(\mathbf{x}) \in \tilde{M}\} \right] = 0. \quad (7.17)$$

From Theorem 3.5 (iii) we have that, for any  $t \in [0, \tau]$ ,

$$\mathbf{T}_{(t)} \circ \mathbf{S}_{(t)}(\mathbf{y}) = \mathbf{y} \quad \text{for } \nu - a.e. \mathbf{y} \in \Lambda.$$

Thus, we have

$$\int_{\tilde{M}} \nu(\mathbf{y}) d\mathbf{y} = 0$$

for any  $t \in [0, \tau)$ . Therefore, using that  $\sigma_0[\widetilde{N}] = 0$ , which implies that  $\mathbf{T}_0 \# \sigma_0 = \nu_0$  for all  $\mathbf{x} \in \Omega \setminus \widetilde{N}$ , and also using (6.13), we obtain for any  $t \in [0, \tau)$

$$\sigma_0 \left[ \{ \mathbf{x} \in \Omega \setminus \widetilde{N} : \Phi_{(t)} \circ \mathbf{T}_0(\mathbf{x}) \in \widetilde{M} \} \right] = \sigma_0 \left[ \{ \mathbf{x} \in \Omega \setminus \widetilde{N} : \mathbf{T}_0(\mathbf{x}) \in \Phi_{(t)}^*(\widetilde{M}) \} \right] \quad (7.18)$$

$$= \int_{\mathbf{T}_0^{-1}(\Phi_{(t)}^{-1}(\widetilde{M}))} \sigma_0(\mathbf{x}) d\mathbf{x} \quad (7.19)$$

$$= \int_{\Phi_{(t)}^{-1}(\widetilde{M})} \nu_0(\mathbf{y}) d\mathbf{y} \quad (7.20)$$

$$= \int_{\widetilde{M}} \nu(\mathbf{y}) d\mathbf{y} = 0,$$

as required.  $\square$

We now show existence of the map  $\mathbf{F}^*$  from Definition 5.2 (iii).

**Proposition 7.6.** *The map  $\mathbf{F}$  as defined in (7.1) satisfies property (iii) of Definition 5.2.*

*Proof.* As with Lemma 7.1, we can show that for every  $t \in [0, \tau)$  the expression  $\mathbf{S}_{(0)} \circ \Phi_{(t)}^* \circ \mathbf{T}_{(t)}(\mathbf{x})$  is defined for  $\sigma - a.e.$   $\mathbf{x} \in \Omega$ , and the map

$$\mathbf{F}_{(t)}^* = \mathbf{S}_{(0)} \circ \Phi_{(t)}^* \circ \mathbf{T}_{(t)}$$

is Borel:

Since  $\mathbf{T}$  exists and is unique  $\sigma - a.e.$  in  $\Omega$  for every  $t \in [0, \tau)$ , we have that  $\mathbf{T}$  exists and is unique on  $\Omega \setminus N^1$  for every  $t \in [0, \tau)$ , where  $N^1$  is a Borel subset of  $\Omega$  with  $\sigma[N^1] = 0$ . Also, since  $\mathbf{S}_{(0)}$  exists and is unique  $\nu_0 - a.e.$  on  $\Lambda$ , we have that  $\mathbf{S}_{(0)}$  exists and is unique on  $\Lambda \setminus N_0^2$ , where  $N_0^2$  is a Borel subset of  $\Lambda$  with  $\nu_0[N_0^2] = 0$ . Then, we have that  $\mathbf{S}_{(0)} \circ \Phi_{(t)}^* \circ \mathbf{T}_{(t)}(\mathbf{x})$  is defined for all

$$\mathbf{x} \in \Omega \setminus (N^1 \cup M),$$

where

$$M = \{ \mathbf{X} \in (\Omega \setminus N^1) : \Phi^*(t, \mathbf{T}_{(t)}(\mathbf{X})) \in N_0^2 \}.$$

Note that, from its definition,  $M$  is a Borel set. We must now show that  $\sigma[M] = 0$  for every  $t \in [0, \tau)$ . Fix  $t \in [0, \tau)$ . Then, using that  $\mathbf{T}_{(t)} \# \sigma = \nu$  and thus  $\mathbf{T}_{(t)} \# \sigma = \nu$  for all  $\mathbf{x} \in \Omega \setminus N^1$ , and using (6.13), we can apply (2.5) and compute

$$\sigma[M] = \sigma \left[ \{ \mathbf{x} \in \Omega \setminus N^1 : \mathbf{T}_{(t)}(\mathbf{x}) \in \Phi_{(t)}(N_0^2) \} \right] \quad (7.21)$$

$$\begin{aligned} &= \int_{\mathbf{T}_{(t)}^{-1}(\Phi_{(t)}(N_0^2))} \sigma(\mathbf{x}) d\mathbf{x} = \int_{\Phi_{(t)}(N_0^2)} \nu(\mathbf{y}) d\mathbf{y} \quad (7.22) \\ &= \int_{N_0^2} \nu_0(\mathbf{y}) d\mathbf{y} = 0. \end{aligned}$$

Thus, we can define  $\mathbf{F}_{(t)}^* = \mathbf{S}_{(0)} \circ \Phi_{(t)}^* \circ \mathbf{T}_{(t)}$  and  $\mathbf{F}^*$  is a Borel mapping.

We can now prove that property (iii) of Definition 5.2 holds. Since  $\mathbf{F}_{(t)} \# \sigma_0 = \sigma$ , we have that  $\mathbf{F}_{(t)}^* \circ \mathbf{F}_{(t)}(\mathbf{x}) = \mathbf{S}_{(0)} \circ \Phi_{(t)}^* \circ \mathbf{T}_{(t)} \circ \mathbf{F}_{(t)}(\mathbf{x})$  for  $\sigma - a.e.$   $\mathbf{x} \in \Omega$ . Then, using Lemma

7.5, we get  $\mathbf{F}_{(t)}^* \circ \mathbf{F}_{(t)}(\mathbf{x}) = \mathbf{S}_{(0)} \circ \Phi_{(t)}^* \circ \Phi_{(t)} \circ \mathbf{T}_0(\mathbf{x})$   $\sigma_0 - a.e.$  in  $\Omega$ . Since, by Lemma 6.1 (iv),  $\Phi_{(t)}^* \circ \Phi_{(t)}(\mathbf{y}) = \mathbf{y}$  for  $\nu - a.e.$   $\mathbf{y}$  and thus for  $\nu_0 - a.e.$   $\mathbf{y} \in \Lambda$ , and since  $\mathbf{T}_0 \# \sigma_0 = \nu_0$ , we have  $\Phi_{(t)}^* \circ \Phi_{(t)} \circ \mathbf{T}_0(\mathbf{x}) = \mathbf{T}_0(\mathbf{x})$  for  $\sigma_0 - a.e.$   $\mathbf{x} \in \Omega$ . Thus,  $\mathbf{F}_{(t)}^* \circ \mathbf{F}_{(t)}(\mathbf{x}) = \mathbf{S}_{(0)} \circ \mathbf{T}_0(\mathbf{x}) = \mathbf{x}$  for  $\sigma - a.e.$   $\mathbf{x} \in \Omega$  by Lemma 3.5 (iii).

By a similar argument, we have that  $\mathbf{F}_{(t)} \circ \mathbf{F}_{(t)}^*(\mathbf{x}) = \mathbf{x}$  for  $\sigma - a.e.$   $\mathbf{x} \in \Omega$ .  $\square$

Finally, we show that property (iv) of Definition 5.2 holds for  $\mathbf{F}$  defined in (7.1).

**Proposition 7.7.** *Let  $\mathbf{F}$  be defined as in (7.1). Then, equality (5.14) holds for any  $\varphi \in C_c^1([0, \tau) \times \Omega; \mathbb{R}^3)$ . Moreover, we have that  $\mathbf{Z}(\cdot, \mathbf{x}) \in W^{1,\infty}([0, \tau))$  for  $\sigma_0 - a.e.$   $\mathbf{x} \in \Omega$ , and (5.21) holds.*

*Proof.* From the definition of the Lagrangian flow  $\Phi$  in Lemma 6.1, we have that

$$\Phi(t, \mathbf{y}) = \mathbf{y} + \int_0^t \tilde{\mathbf{w}}(s, \Phi_{(s)}(\mathbf{y})) ds$$

for  $\nu_0 - a.e.$   $\mathbf{y} \in \Lambda$  and every  $t \in [0, \tau)$ . Thus, this equality holds for all  $\mathbf{y} \in \Lambda \setminus N$  where  $\nu_0[N] = 0$ . Since  $\mathbf{T}_0 \# \sigma_0 = \nu_0$  it follows that

$$\sigma_0 [\Omega \cap \mathbf{T}_0^{-1}(N)] = \int_{\mathbf{T}_0^{-1}(N)} \sigma_0(\mathbf{x}) d\mathbf{x} = \int_N \nu_0(\mathbf{y}) d\mathbf{y} = 0.$$

Thus, for  $\sigma_0 - a.e.$   $\mathbf{x} \in \Omega$  and every  $t \in [0, \tau)$ , we have

$$\Phi(t, \mathbf{T}_0(\mathbf{x})) = \mathbf{T}_0(\mathbf{x}) + \int_0^t \mathbf{w}(s, \Phi_{(s)}(\mathbf{T}_0(\mathbf{x}))) ds, \quad (7.23)$$

where we have replaced  $\tilde{\mathbf{w}}(s, \Phi_{(s)}(\mathbf{T}_0(\mathbf{x})))$  by  $\mathbf{w}(s, \Phi_{(s)}(\mathbf{T}_0(\mathbf{x})))$  based on (6.6), (6.7). Multiplying (7.23) by  $\sigma_0(\mathbf{x})$  and by  $\partial_t \varphi(t, \mathbf{x})$ , where  $\varphi \in C_c^1([0, \tau) \times \mathbb{R}^3)$ , and then integrating we obtain

$$\begin{aligned} \int_{[0, \tau) \times \Omega} \partial_t \varphi(t, \mathbf{x}) \Phi(t, \mathbf{T}_0(\mathbf{x})) \sigma_0(\mathbf{x}) dt d\mathbf{x} &= \int_{[0, \tau) \times \Omega} \partial_t \varphi(t, \mathbf{x}) \mathbf{T}_0(\mathbf{x}) \sigma_0(\mathbf{x}) dt d\mathbf{x} \\ &+ \int_{[0, \tau) \times \Omega} \sigma_0(\mathbf{x}) \partial_t \varphi(t, \mathbf{x}) \int_0^t \mathbf{w}(s, \Phi_{(s)}(\mathbf{T}_0(\mathbf{x}))) ds dt d\mathbf{x}. \end{aligned} \quad (7.24)$$

Now, in the right-hand side, we perform the integration with respect to  $t$  in the first integral and integrate by parts with respect to  $t$  in the second integral to obtain

$$\int_{[0, \tau) \times \Omega} \partial_t \varphi(t, \mathbf{x}) \Phi(t, \mathbf{T}_0(\mathbf{x})) \sigma_0(\mathbf{x}) dt d\mathbf{x} = - \int_{\Omega} \varphi(0, \mathbf{x}) \mathbf{T}_0(\mathbf{x}) \sigma_0(\mathbf{x}) dt d\mathbf{x} \quad (7.25)$$

$$- \int_{[0, \tau) \times \Omega} \varphi(t, \mathbf{x}) \mathbf{w}(t, \Phi_{(t)}(\mathbf{T}_0(\mathbf{x}))) \sigma_0(\mathbf{x}) dt d\mathbf{x}, \quad (7.26)$$

where we have used that  $\varphi(\tau, \mathbf{x}) \equiv 0$  due to its compact support. Note that, by (3.15), (7.1) and Lemma 7.5, we have

$$\mathbf{w}(t, \Phi_{(t)}(\mathbf{T}_0(\mathbf{x}))) = f_{\text{cor}} \mathbf{e}_3 \times [\Phi_{(t)}(\mathbf{T}_0(\mathbf{x})) - \mathbf{S}_{(t)}(\Phi_{(t)}(\mathbf{T}_0(\mathbf{x})))] = f_{\text{cor}} \mathbf{e}_3 \times [\mathbf{Z}(t, \mathbf{x}) - \mathbf{F}(t, \mathbf{x})] \quad (7.27)$$

for  $\sigma_0 - a.e. \mathbf{x} \in \Omega$  and every  $t \in [0, \tau)$ . Substituting (7.27) into the right-hand side of (7.26) and using Lemma 7.5 to replace  $\Phi(t, \mathbf{T}_0(\mathbf{x}))$  by  $\mathbf{Z}(t, \mathbf{x})$  in the left-hand side of (7.26), we obtain

$$\begin{aligned} \int_{[0, \tau) \times \Omega} \partial_t \varphi(t, \mathbf{x}) \mathbf{Z}(t, \mathbf{x}) \sigma_0(\mathbf{x}) dt d\mathbf{x} &= - \int_{\Omega} \varphi(0, \mathbf{x}) \mathbf{T}_0(\mathbf{x}) \sigma_0(\mathbf{x}) dt d\mathbf{x} \\ &\quad - \int_{[0, \tau) \times \Omega} \varphi(t, \mathbf{x}) f_{\text{cor}} \mathbf{e}_3 \times [\mathbf{Z}(t, \mathbf{x}) - \mathbf{F}(t, \mathbf{x})] \sigma_0(\mathbf{x}) dt d\mathbf{x}, \end{aligned} \quad (7.28)$$

and rearranging gives (5.14).

Finally,  $\mathbf{Z}(\cdot, \mathbf{x}) \in W^{1, \infty}([0, \tau))$  for  $\sigma_0 - a.e. \mathbf{x} \in \Omega$  follows from Lemma 7.5 and Lemma 6.1 (i). Then, (7.23), (7.27) and Lemma 7.5 imply (5.21).  $\square$

Now the properties of  $(\mathbf{T}, \sigma)$  in Theorem 4.1 and the properties of  $\mathbf{F}$  proved in Propositions 7.2, 7.3, 7.4, 7.6, 7.7 imply Theorem 5.5.

## 8 Conclusion

The main result of this paper is the proof of existence of weak Lagrangian solutions of the fully compressible semi-geostrophic equations with rigid boundary conditions, in the original formulation with variables expressing physically relevant quantities. This result is stated in Theorem 5.5, and can be considered as the conclusion of the analysis of the problem given in [8], where an existence result is proved but only in the so-called dual formulation. We have also proved that, if additional regularity of the flow could be assumed, this weak Lagrangian solution would determine a weak (Eulerian) solution of these equations.

In addition to the main result, we have given an alternative proof, based on recent results of Ambrosio and Gangbo on Hamiltonian ODEs in spaces of probability measures, of the previous result on the existence of weak solutions of the dual formulation of the equations.

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